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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled DIRECT AND FINITE VARIATIONAL METHODS FOR THE ELASTIC WAVE EQUATION submitted by YVES CLAUDE ATHIAS in partial fulfilment of the requirements for the degree of DOCTOR OF PHILOSOPHY in PHYSICS.

ABSTRACT

The aim of this thesis is to have a closer look at classical numerical analysis as a tool for solving the elastic wave equation in its general case.

After topological considerations of the parameter distributions at the boundary, and their relationship to the grid and the operator domains, we develop numerical schemes by the direct method of the numerical analysis (i.e. the differential equation and the external conditions). We also examine the variational or energy method where the boundary conditions are naturally introduced by Gauss' Theorem in the potential energy equation. For this purpose we will develop also a method of discretisation of the Dirichlet integral, and its bilinear form. Comparisons between those schemes show the uniqueness of the development. A finite difference example illustrates the P - SV conversions.

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CHAPTER I

Introduction

The properties of body and surface waves have been investigated by many researchers. More recently, numerical analysis has been a powerful tool. In their celebrated paper, Courant, Friedrichs and Lewy (1928) analysed the finite difference method but it is only after numerical techniques have been used for some time for differential equations of physics and engineering that they have been applied to hyperbolic and transient elastic wave problems (Alterman et al, 1968). Alterman and coworkers studied the wave propagation in a fluid sphere (1968), in a layered elastic media (1968), in an elastic sphere (1970), and in a quarter and three quarter plane (1970).

Boore (1970) investigated the SH and Love wave propagations in the transition region from ocean to continent. Landers and Claerbout (1972) investigated the direct problem of the potential wave equation in elastic media. Claerbout developed the parabolic approximation of the scalar wave equation for direct and migration problems. Finite difference methods using finite integral transformations of one or more variables have been widely used (Soltz, 1978).

Boore (1972) studied SH wave propagation in hetero-

geneous media and used successfully an equation with variable parameters. This method had an instant success since cumbersome boundary conditions are no longer needed and arbitrary boundaries could be considered without adding any complexity to the problem. This method, called "heterogeneous media", was soon extended to the elastic wave equation (Kelly et al, 1976). Finite element methods (Zienkiewicz, 1971) have been investigated by Lysemer and Drake (1972) for the direct problem and by Marfurt (1977) for the elastic wave equation migration problem.

Recently the necessity of obtaining the maximum information from the data in seismic exploration has given renewed interest in P-SV and SH waves and consequently to the elastic wave equation. The first aim of a researcher is therefore to develop a more efficient and optimized algorithm. Among the methods, of course, the "heterogeneous media" is the most attractive. The inconsistency in the results and the disparity with the "homogeneous media" method (Kelly et al, 1976) has brought some doubt on this technical legacy. Then, considering the "experimental" aspect of the so-called "homogeneous media" method (Chapter II) one wonders if it is not necessary to solve the problem before computation, rather than to input the equation with external conditions in the machine and carry on endless trials and comparisons.

The objective is therefore to find a scheme where the boundary conditions are well defined in any point of a considered region for any boundary shape, provided of course that those boundaries pass by the grid knots. For this purpose the classical numerical analysis offers the choice between the direct method and the method based on energy or variational principles (Chapter II).

As in any discrete problems we shall consider an elementary domain and iterate the solution in all the considered regions. The direct approach will be obtained through a Taylor series development, the boundary conditions being introduced in the equations (method developed in Chapter III).

It will then be interesting to find a solution by the finite variational method, since the boundary conditions are naturally included in the integral equation (Chapter IV). A comparison with the direct approach can give us a new insight into the problem as well as on the validity of the solutions obtained.

Since the source of P-SV conversion is an important question which can be more clearly analysed through the divergence and the curl of the displacement vector (Chapter IV) we will investigate those equations under two aspects:

- Compatibility of the equations with the displacement solution will give a greater insight

into different aspects of the problem.

- Comparison with previous work.

The results of this research have been presented by the author at the 50th Society of Exploration Geophysicists Convention (Houston, 1980).

CHAPTER II

NUMERICAL FORMULATION

2.1 Equations of Motion

The equations of motion can be deduced from the principle of minimum potential energy in elasticity.

Let u_i be the components of the displacement vector U

σ_{ij} the stresses components

K_i the body forces components per unit volume

e_{ij} the strain components

ρ the density.

The deformation components are related to the displacement vector by

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \qquad (2.1.1)$$

The medium is elastic when there exist a linear relationship between stresses and strain (Generalised Hooke's law, Love, 1927)

$$\sigma_{ij} = c_{ijk\ell}\, e_{k\ell} \quad (i,j,k,\ell = 1,2,3) \qquad (2.1.2)$$

$c_{ijk\ell}$ being the elastic constants with

$$\sigma_{ij} = \sigma_{ji}$$

so that

$$c_{ijk\ell} = c_{k\ell ij}$$

This follows from the existence of an elastic potential

$$2W = c_{ijk\ell}\, e_{ij}\, e_{k\ell} \qquad (2.1.3)$$

such that

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}} \qquad (2.1.4)$$

From (2.1.2) and (2.1.3) we obtain

$$2W = e_{11}\sigma_{11} + e_{22}\sigma_{22} + e_{33}\sigma_{33} + e_{12}\sigma_{12} + e_{13}\sigma_{13} + e_{23}\sigma_{23} \qquad \ldots(2.1.5)$$

We shall represent the elastic potential by W(U). The relation (2.1.3) shows that W is a quadratic form in the strain components. In elasticity theory it is proved that this form is positive bounded below (Miklin, 1967).

If the elastic medium is isotropic then the Hooke's law reduces to the Lamé equations

$$\sigma_{ij} = \lambda\theta\delta^{i}_{j} + \mu e_{ij} \qquad (2.1.6)$$

with $\theta = \operatorname{div} U$, δ^{i}_{j} the Krokener delta. λ, μ are the Lamé constants.

Then, the equilibrium equations can be written in the form

$$-D_j\sigma_{ij} = F_i \tag{2.1.7}$$

where $D_j = \frac{\partial}{\partial x_j}$

$F_i = -\rho D_{tt}u_i + K_i$

K_i being the body forces per unit volume.

And from (2.1.7) we have

$$AU = F \tag{2.1.8}$$

where the operator A, in the case of an isotropic medium, is:

$$A \equiv (\lambda + 2\mu)\nabla(\nabla.) - \mu\nabla x(\nabla x) \tag{2.1.9}$$

When the parameters are variables, (2.1.8) can be written

$$\rho D_{tt}u_i - D_i[\lambda D_j u_j + 2\mu D_i u_i]$$
$$- D_j[\mu D_j u_i + D_i u_j] - K_i = 0 \tag{2.1.10}$$

or in vectorial form (Karal and Keller, 1959)

$$\rho\ddot{\vec{u}} - \nabla[(\lambda + 2\mu)\nabla.\vec{u}] + \nabla x[\nabla x\vec{u}] - \vec{K} = 0 \tag{2.1.11}$$

Those equations are expressed in a variable parameter form, since this form is more general and coincides with equation (2.1.7).

If the propagation occurs in the (x,z) plane the components of $\vec{u}$ are u,w and the P-SV equations of motion are:

$$\rho D_{tt}u = D_x[\lambda[D_x u + D_z W] + 2\mu D_x u]$$

$$+ D_z[\mu[D_x W + D_z u]] \qquad (2.1.12)a$$

$$\rho D_{tt}w = D_z[\lambda[D_z w + D_x u]] + 2\mu D_z w]$$

$$+ D_x[\mu[D_z u + D_x w]] \qquad (2.1.12)b$$

and for the SH waves

$$\rho D_{tt}V = D_x(\mu D_x V) + D_z(\mu D_z V) \qquad (2.1.13)$$

where the vibration is perpendicular to (x,z).

The systems (2.1.12) and (2.1.13) are independent of each other. ρ, λ, μ can be constant in a region (Ω) or represented by a continuous differentiable function. In both cases we have $\rho, \lambda, \mu, \varepsilon\ (\mathbb{C}^1)$. Besides these systems are submitted to auxiliary conditions which are:

- The displacement and velocity fields at the initial instant.

- The continuity of displacement and stresses at the boundary (Γ) (i.e. when $\rho, \lambda, \mu \not\in \mathbb{C}^1$) (fig. 1, page 14). We are therefore led to the following definitions.

A homogeneous medium is a region R ($R \subset \mathbb{R}^n$) where the parameters (or the physical properties of the media) are constants.

When the medium admits continuous derivatives at each point $P \in R$, $(\lambda, \mu, \ell) \in \mathbb{C}^1$, then we will consider the medium as a transition zone. (Example: variation of velocity with depth, etc.).

A heterogeneous medium is a region R where the physical properties do not admit smooth variations in at least one of its points P. (i.e. $\exists$ a point $P \in R$ such that ρ, λ or $\mu \notin \mathbb{C}^1$). Consequently this point is an element of the boundary Γ.

Very often heterogeneous media are assimilated to transition zones by simple reason of convenience.

2.2 BASIC CONSIDERATIONS

2.2.a Physical laws governing the system

In this section as well as in section 2.3, we will try to state in the most concise terms the broad lines of any correct numerical solution. Although it may seem too general, it will constitute the base on which our problems will be solved.

Let us consider an n dimensional Euclidean space $\mathbb{R}^n$ and a real time interval $[0,T]$.

If Ω is a region of $\mathbb{R}^n$ with boundary Γ, we have

$$\overline{\Omega} = \Omega + \Gamma$$

U is the displacement vector defined in the space of scalar or vector functions, respectively defined in the set of points

$$\overline{\Omega} \times [0,T]$$

or $$\Omega \times [0,T] \text{ and } \Gamma \times [0,T]$$

The equations of motion (2.1.8) can be written in the more concise form:

$$AU(P) + F = 0$$

$$P \varepsilon \bar{\Omega} \times [0,T]$$

$$U, \partial U/\partial x_i \ \varepsilon \ L^2(\Omega) \qquad (2.2.1)a$$

where A is the elasticity operator

and $$F = \rho D_{tt}U - K(P) \qquad (2.2.1)b$$

K(P) representing the body forces per unit volume at P.
The subsidiary conditions being:

- The initial conditions

$$U(x,0) = f(x,0)$$

$$\dot{U}(x,0) = g(x,0) \qquad (2.2.2)$$

(The initial conditions being homogeneous if the "source" is introduced in (2.1.1) as Body force).

- The boundary conditions

Generally they are classed as the following:

(a) The boundary Γ is fixed

$$U\Big|_{\Gamma} = 0 \qquad (2.2.3)a$$

(b) Γ is free from external forces

$$\sigma_{ij} \cdot \vec{n} \Big|_{\Gamma} = 0 \qquad (2.2.3)b$$

(c) A part of the boundary is fixed, and the other is "free".

In the wave propagation problem we will be mainly concerned with the conditions (b) at the surface and with the continuity conditions at the boundary

$$U \Big|_{\Gamma^-} = U \Big|_{\Gamma^+} \quad \text{(displacement continuity)} \qquad (2.2.3)$$

$$\sigma_{ij} \cdot \vec{n} \Big|_{\Gamma^-} = \sigma_{ij} \cdot \vec{n} \Big|_{\Gamma^+} \quad \text{(stress continuity)} \qquad (2.2.4)$$

where Γ^-, Γ^+ represents the interior or exterior part of the boundary.

If $\vec{n}$ is the outward normal to the boundary the stress continuity (2.2.4)b can have the general form (Mikhlin, 1964)

$$\sum_{i,j} a_{ij} \, D_j \, U \cos(n, x_i) \Big|_{\Gamma^+} = \sum_{i,j} a_{ij} \, D_j \, U \cos(n, x_i) \Big|_{\Gamma^-} \qquad \ldots (2.2.5)$$

or

$$\sum_{i,j} a_{ij} \, D_{\vec{n}} U \Big|_{\Gamma} = 0 \qquad (2.2.6)$$

where $D_{\vec{n}}U = \frac{DU}{\partial n}$ is the normal derivative to the surface boundary Γ, the sign (+) being attributed to the exterior region of the boundary and the sign (-) to the interior. The condition (2.2.6) can be considered as a generalized Neumann condition.

2.2.b The energy method

If we construct the inner product

$$(AU,U) = \int_{\Omega} U.AUd\Omega \tag{2.2.7}$$

we have (Betti formula)

$$(AU,U) = 2\int_{\Omega} W(U)d\Omega - \int_{\Gamma} U\tau^{(n)}d\Gamma \tag{2.2.8}$$

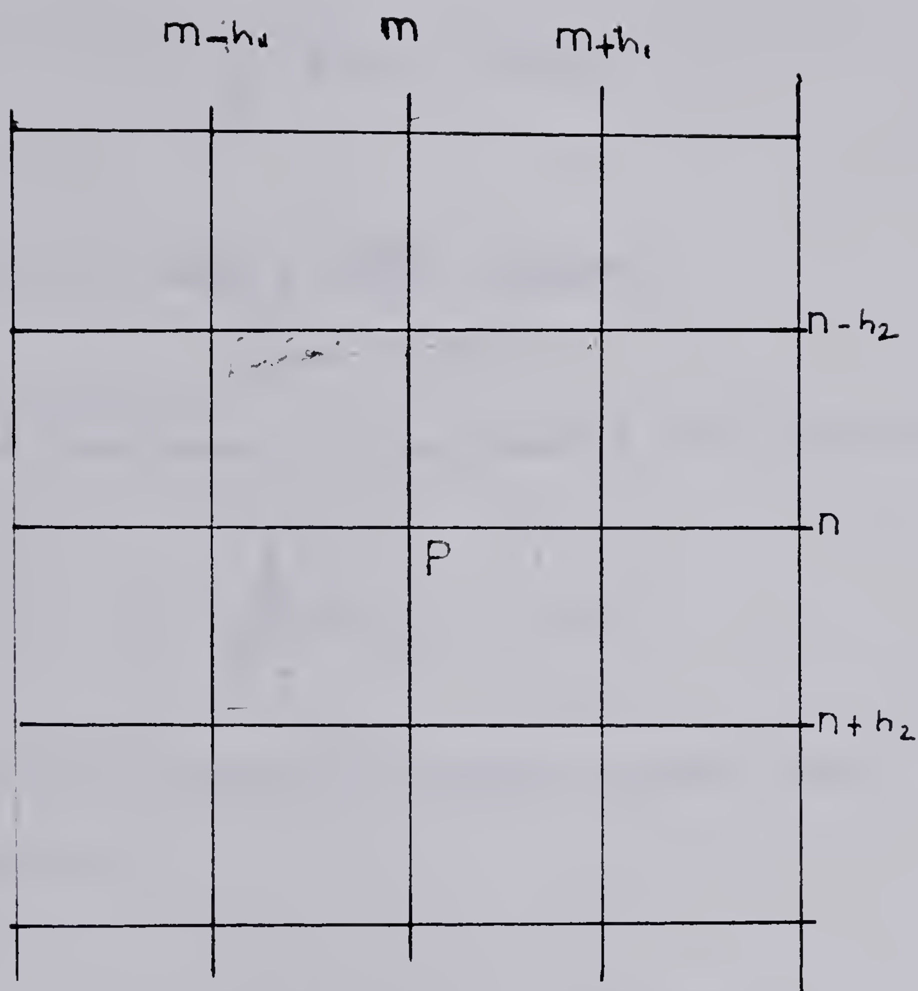
where $\tau^{(n)}$ is the stress vector acting on the surface of the boundary.

Since

$$\int_{\Gamma} U\tau^{(n)}d\Gamma = 0 \quad \text{(under the boundary conditions 2.2.4)}$$

then,

$$(AU,U) = 2\int_{\Omega} W(U)d\Omega > 0 \tag{2.2.9}$$



$$D = \Omega_n = P(m,n) \pm h_i$$

Fig. 2.1. Representation of an elementary region

Thus the operator A is positive definite[1]. The problem enumerated can be shown to be equivalent to finding the minimum of the potential energy

$$\int_{\Omega} (W(U) - UF)\,d\Omega \tag{2.2.10}$$

i.e.
$$\min \int_{\Omega} \left(\frac{UAU}{2} - UF\right)d\Omega \tag{2.2.11}$$

which can be expressed as minimizing the functional

$$I = \frac{1}{2}(AU,U) - (U,F) \tag{2.2.12}$$

which is the Dirichelet principle (Lions 1970) which states that

$$\text{Inf } I(v) = I(U) \qquad U \in H'(\Omega) \tag{2.2.13}$$

The solution $v \in H'(\Omega)$ being unique, and where $H'(\Omega)$ is the space of finite energy (Sobolev space, W^1_2; Sobolev, 1953) which is Hilbert space with norm equal to

---

(1) Note: this property is obvious in the case of SH wave where $A = -\nabla^2$

then
$$-\int_{\Omega} U\nabla^2 U d\Omega = \int_{\Omega} (\nabla U)^2 d\Omega - \int_{\Gamma} \frac{\partial U}{\partial n}\, dB = 0$$

and
$$(AU,U) = (U,AU) > 0$$

$$\left\| v \right\|_{H'(\Omega)} = \left[\left\| v \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} \right]^{\frac{1}{2}} \qquad (2.2.14)$$

We note that (2.2.13) is the extension of the Dirichlet principle to a more general class of operators and boundary conditions.

2.3 PRINCIPLE OF NUMERICAL SOLUTION

2.3. Generalities

The general principle is of course to operate in a finite dimensional space.

Let us consider the open region $\Omega \subset \mathbb{R}^n$ of boundary Γ and a basis $W_1, \ldots, W_m$ in $H'(\Omega)$ such that

- $W_1, \ldots, W_m$ are linearly independent $\forall m$
- The linear finite combination

$$\sum \xi_j W_j \quad , \ \xi_j \in R$$

are dense in $H'_o(\Omega)$.

The problem has three equivalent solutions (Lions,1970)

(i) the usual formulation given by (2.2.1), (2.2.4).

(ii) a Dirichlet principle formulation, the solution U_m being given by

$$\operatorname*{Inf}_{V \in W_m} I(V) = I(U_m) \ , \ U_m \in W_m \qquad (2.3.1)$$

which is the Ritz method.

(iii) or by the equivalent form

$$(A\ U_m, W) = (f, W) \quad \forall\ W\ \varepsilon\ W_m \tag{2.3.2}$$

$$\forall\ P\ \varepsilon\ (\overline{\Omega}) = \Omega + \Gamma \subset \mathbb{R}^n$$

which is the Galerkin method.

We note that the advantage of form (2.1.1) and (2.1.2) is that the boundary conditions are included in the equation to solve (see note p. 30).

Then, the main questions raised are:

(i) the convergence and stability

(ii) the choice of the base $(W_q, \ldots, W_m)$

The second question being subject to the matrix $\| A(W_i, W_j) \|$ being sparse which implies that

$$\{\text{support } W_i\} \cap \{\text{support } W_j\} = \emptyset$$

where $\{W_1, \ldots, W_m\}$ are functions characteristic of separable sets such that $\{W_q, \ldots, W_m\}$ are dense in $H'(\Omega)$

and $$\forall\ W_i\ \varepsilon\ H'(\Omega) \subset L^2(\Omega)$$

We will use those properties in Chapter III to develop finite difference schemes in which boundary conditions are implied.

2.3.b Well posed problem

From the above section we can say the problem consists in the mapping of the initial data to the space of admitted solution U.

Referring to Hadamard (1953) the problem is well posed if

(i) to every set of initial data corresponds one and only one admitted solution U. (i.e. A has a unique inverse A^{-1}).

(ii) the solution U depends continuously on the initial data which implies

- If $\tilde{U}_o$ is an initial state which is not element of the analytical solution, there exists an element U_o of the solution such that $|\tilde{U}_o - U_o| < \varepsilon$

- The stability (as a consequence of the continuity).

Besides, the geophysical requirements are:

- To handle arbitrary admitted data

- The solution has to satisfy external constraints, i.e. initial conditions and arbitrary boundaries.

- The error truncation has to be well defined.

2.4 FINITE DIFFERENCE SCHEME

2.4.1.a

The finite difference schemes are given by the discretization of the operators A and D_{tt}.

We first note that the decomposition of A has the form

$$A = \sum_{i,j} a_{ij} D_{ij} \tag{2.4.1}$$

where

$$D_{ij} = D_{x_i} D_{x_j} \tag{2.4.2}$$

Let us consider in the plane (x,z) a grid of period (h_1, h_2) and a point P(x,z,t) such that at a given instant $t = \ell\Delta t$. We have (fig. 2)

$$P(x,z,t) = P^{\ell\Delta t}(mh_1, nh_2) = P^{\ell}_{m,n} \tag{2.4.3}$$

By a simple Taylor expansion we obtain the spatial discretization (i.e. semi discretization)

$$D_{xx}U = \frac{1}{h_1^2} [U_{m-1,n} - 2U_{m,n} + U_{m+1,n} - \frac{1}{12} (h_1^4 D_x^4 U)] \tag{2.4.4}$$

or

$$D_{xx}U = \tilde{D}_{xx}U - \frac{1}{12h_1^2} (h_1^4 D_x^4 U) \tag{2.4.5}$$

with

$$\tilde{D}_{xx}U_{m,n} = U_{m-1,n} - 2U_{m,n} + U_{m+1,n} \tag{2.4.6}$$

or with the notation in figure 2.2, page 24

$$\tilde{D}_{xx}U_0 = \frac{1}{h_1^2} (U_3 - 2U_0 + U_1) \tag{2.4.7}$$

By the same way

$$D_{zz}U_{m,n} = \frac{1}{h_2^2}\,[U_{m,n-1} - 2U_{m,n} + U_{m,n+1}] - \frac{1}{12h_2^2}\; h_2^4\, D_z^4\, U$$

$$D_{zz}U = \tilde{D}_{zz}U_{m,n} - \frac{1}{12h_2^2}\,[h^4\, D_z^4\, U] \qquad (2.4.8)$$

with

$$\tilde{D}_{zz}U = \frac{U_4 - 2U_0 + U_2}{h_2^2} \qquad (2.4.9)$$

and

$$\tilde{D}_{xz}U = \frac{1}{4h_1h_2}\,[U_{m+1,n+1} + U_{m-1,n-1} - U_{m+1,n-1} - U_{m-1,n+1}]$$

$$- \frac{1}{4h_1h_2}\,[h_1^2\, h_2^2\, D_x^2\, D_z^2\, U] \qquad (2.4.10)$$

$$= \tilde{D}_{xz}U - \frac{1}{4h_1h_2}\,[h_1^2\, h_2^2\, D_x^2\, D_z^2\, U] \qquad (2.4.11)$$

with

$$\tilde{D}_{xz}U = \frac{1}{4h_1h_2}\,[U_6 + U_8 - U_5 - U_7] \qquad (2.4.12)$$

We note that the second order finite difference operators $\tilde{D}_{xx}$, $\tilde{D}_{zz}$ are convolution operators of the form (1, -2, 1) which lead to the well known Crank-Nicolson tridiagonal form (Mitchel, 1978; Richtmeyer and Morton, 1967).

By the same way we have for the time variable

$$D_{tt}U = (U^{\ell-1} - 2U^{\ell} + U^{\ell+1}) \frac{1}{\Delta t^2} - \frac{1}{12\Delta t^2} [\Delta t^4 \; D_t^4 \; U] \quad (2.4.16)$$

$$D_{tt}U = \tilde{D}_{tt}U - \frac{1}{12\Delta t^2} [\Delta t^4 \; D_t^4 \; U] \quad (2.4.17)$$

with

$$\tilde{D}_{tt}U = \frac{U^{-1} - 2U + U^{+1}}{\Delta t^2} \quad (2.4.18)$$

2.4.1.b Consistancy of the operator form

From (2.4.1) we can write

$$A = \tilde{A} + \sum \frac{a_{ij}}{h_i h_j} (h_i^2 \; h_j^2 \; D_i^2 \; D_j^2) \quad (2.4.19)$$

since the operator D_j^2 is positive definite (2.2.b)

$$\frac{a_{ij}}{h_i h_j} [h_i^2 \; h_j^2 \; D_i^2 \; D_j^2 \; U] \geq 0 \quad (2.4.20)$$

Then $\tilde{A}$ is a lower bound of A when $h_1, h_2 \rightarrow 0$. Consequently if $\tilde{A}$ is a closed positive definite bounded below operator so is A. To check the symmetry in Ω it suffices to write (2.1.11) in the form

$$- \begin{pmatrix} [(\lambda + 2\mu)\tilde{D}_{xx} + \mu_{ij}\tilde{D}_{zz})] & (\lambda + \mu)\tilde{D}_{xz} \\ (\lambda + \mu)\tilde{D}_{xz} & (\lambda + 2\mu)\tilde{D}_{zz} + \mu\tilde{D}_{xx} \end{pmatrix} \vec{U}(P)$$

$$= -\rho \; D_{tt} \; \vec{U}(P) \qquad (2.4.21)$$

where $P(mn_1, mn_2) \in \Omega_h \subset \Omega$ (fig. 2)

where Ω_h is the space spanned by $P_{m,n}$ and its neighborhood.

We see that the discretization from $A \to \tilde{A}$ conserve the symmetry property of $A \; \forall \; U_i(P)$, $P \in (\Omega_h)$ when P is a point of coordinate $m_1h_1, \ldots, m_ih_i$ and Ω_h is the subdomain defined by $P \pm h_i$.

(2.1.1) yields

$$A_hU_h + D_{tt}U_h = K_h^{\ell} \qquad (2.4.22)$$

we are then led to the following methods

$$(A_hU_h^{\ell+1}, U_h) + \frac{U_h^{\ell+1} - 2U_h^{\ell} + U_h^{\ell-1}}{\Delta t^2}, U_h = (K^{\ell+1}, U_h) \; (2.4.23)$$

or

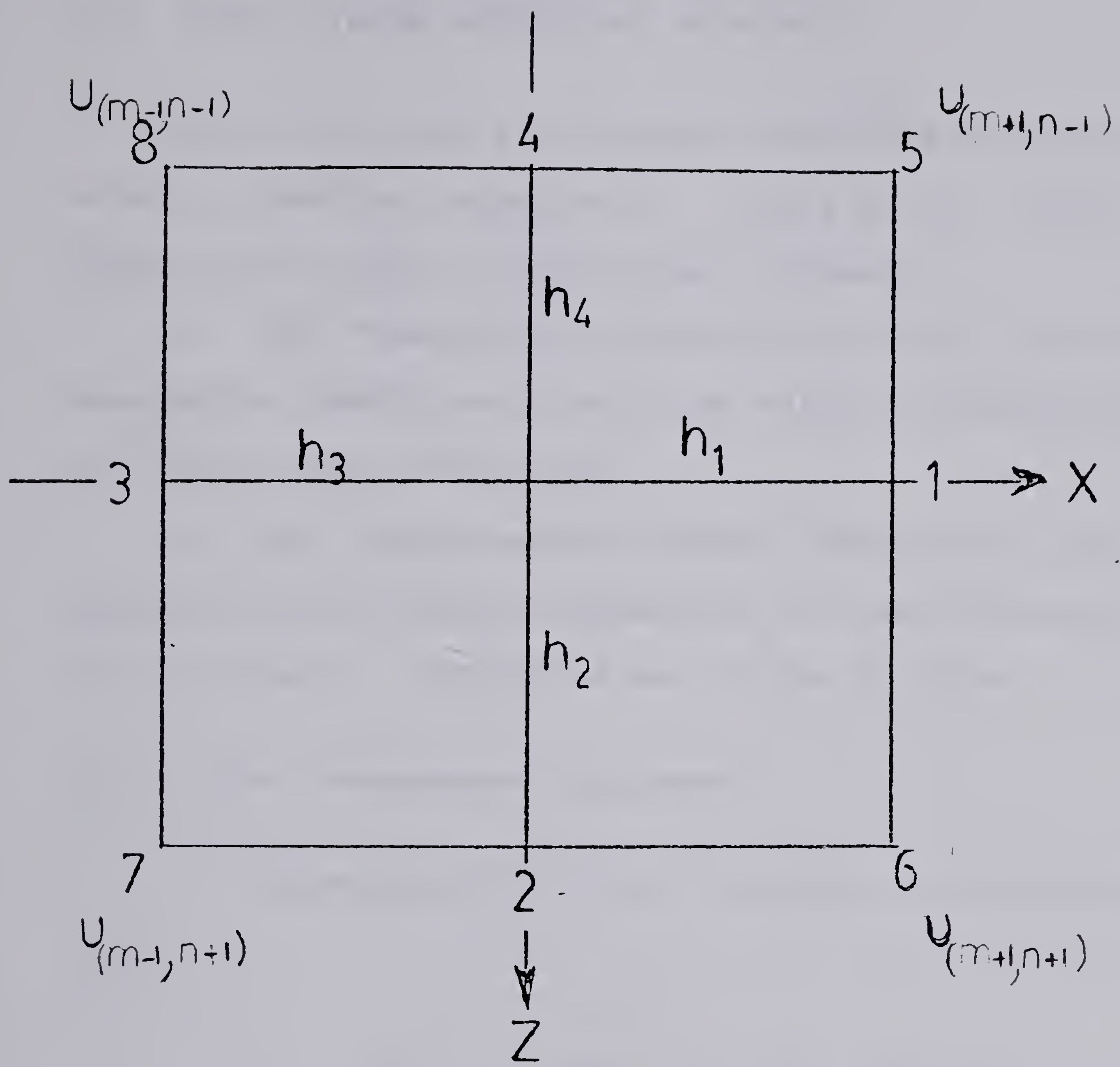
$$(A_hU_h, v_h) + \frac{U_h^{\ell+1} - 2U_h^{\ell} + U_h^{\ell-1}}{\Delta t^2}, v_h = (K^{\ell}, v_h) \qquad (2.4.24)$$

where U_h^{ℓ} and $U_h^{\ell+1}$ represent the displacement $U(P)$ $P \in \Omega_h$ at the instant $\ell\Delta t$ and $(\ell+1)\Delta t$ or more succintly

$$P \in \Omega_h \subset \mathbb{R}^n \times [0T)$$

The scheme (2.4.23) is an implicit scheme, the scheme (2.4.24) being explicit.

The implicit scheme is unconditionally stable (Mitchell, 1978) (see Section 3-10).



$U = U_{m,n} \qquad U_1 = U_{m+1,n} \quad U_2 = U_{m,n+1}$ - - - - - - -

fig. 2.2

Lattice used in finite difference representation

2.5 USUAL FINITE DIFFERENCE APPROACH

The usual finite difference methods are mostly evaluated by numerical experiments. Kelly et al. (1976) investigated the two main computational schemes.

i) The "homogeneous formulation" where the standard boundaries conditions have to be satisfied between the two media being considered.

ii) The "heterogeneous media" formulation where the equation with variable parameters is used and where the only boundary prescribed are at the surface.

2.5.a The "homogeneous approach"

Equations (2.1.12) are written in the following form:

$$\tilde{D}_{tt}U = \alpha^2\tilde{D}_{xx}U + \beta^2\tilde{D}_{zz}U + (\alpha^2 - \beta^2)\tilde{D}_{zx}W$$

$$\tilde{D}_{tt}W = \alpha^2\tilde{D}_{zz}W + \beta^2\tilde{D}_{xx}W + (\alpha^2 - \beta^2)\tilde{D}_{zx}U \qquad (2.5.1)$$

The B.C. are for a horizontal interface

$$W^+ = W^- \quad ; \quad U^+ = U^-$$

(Displacement continuity)

$$\tilde{\sigma}^{+}_{zz} = \tilde{\sigma}^{-}_{zz} \quad ; \quad \tilde{\sigma}^{+}_{xz} = \sigma^{-}_{xz}$$

(Stress continuity)

with $\tilde{\sigma}_{zz} = 0$; $\tilde{\sigma}_{xz} = 0$ at the surface. (2.5.2)

The $\sim$ represents the finite difference expansion, and

$$\tilde{\sigma}_{zz} = \alpha^2 \tilde{W}^2_z + (\alpha^2 - 2\beta^2)\tilde{U}_x$$

$$\tilde{\sigma}_{xz} = \beta^2 [\tilde{W}_x + \tilde{U}_z]$$

Then the systems (2.5.1), (2.5.2) are applied to each interface, using as expansion of media approach the fictitious lines (fig. 2.3). For details of the method, see Alterman and Karal (1968).

2.5.b The "heterogeneous media approach"

The equation (2.1.12) are written taking into account that the coefficient are variable, We then obtain

$$\tilde{D}_{tt}U = \alpha^2 D_{xx}U + \alpha^2_x \tilde{D}_x U + \beta^2 \tilde{D}_{zz}U + \beta^2_x \tilde{D}_x U + (\alpha^2 - 2\beta^2)_x \tilde{D}_z W$$

$$+ \beta^2_z \tilde{D}_x W \qquad (2.5.3)$$

$$\tilde{D}_{tt}W = \alpha^2\tilde{D}_{zz}W + \alpha_2^2\tilde{D}_z W + \beta^2\tilde{D}_{zz}W + \beta_x^2\tilde{D}_{zz}W + (\alpha^2-2\beta^2)_z\tilde{D}_x U$$

$$+ \beta_x^2 D_z U \qquad (2.5.3)$$

surface conditions being expressed by

$$\sigma_{xz} = 0$$

$$(2.5.4)$$

$$\sigma_{xz} = 0$$

with the same procedure of fictitious line than the above-mentioned.

We see that the tentative solution consists in ignoring the boundary conditions by assimilating the boundaries to a limit of transition zone.

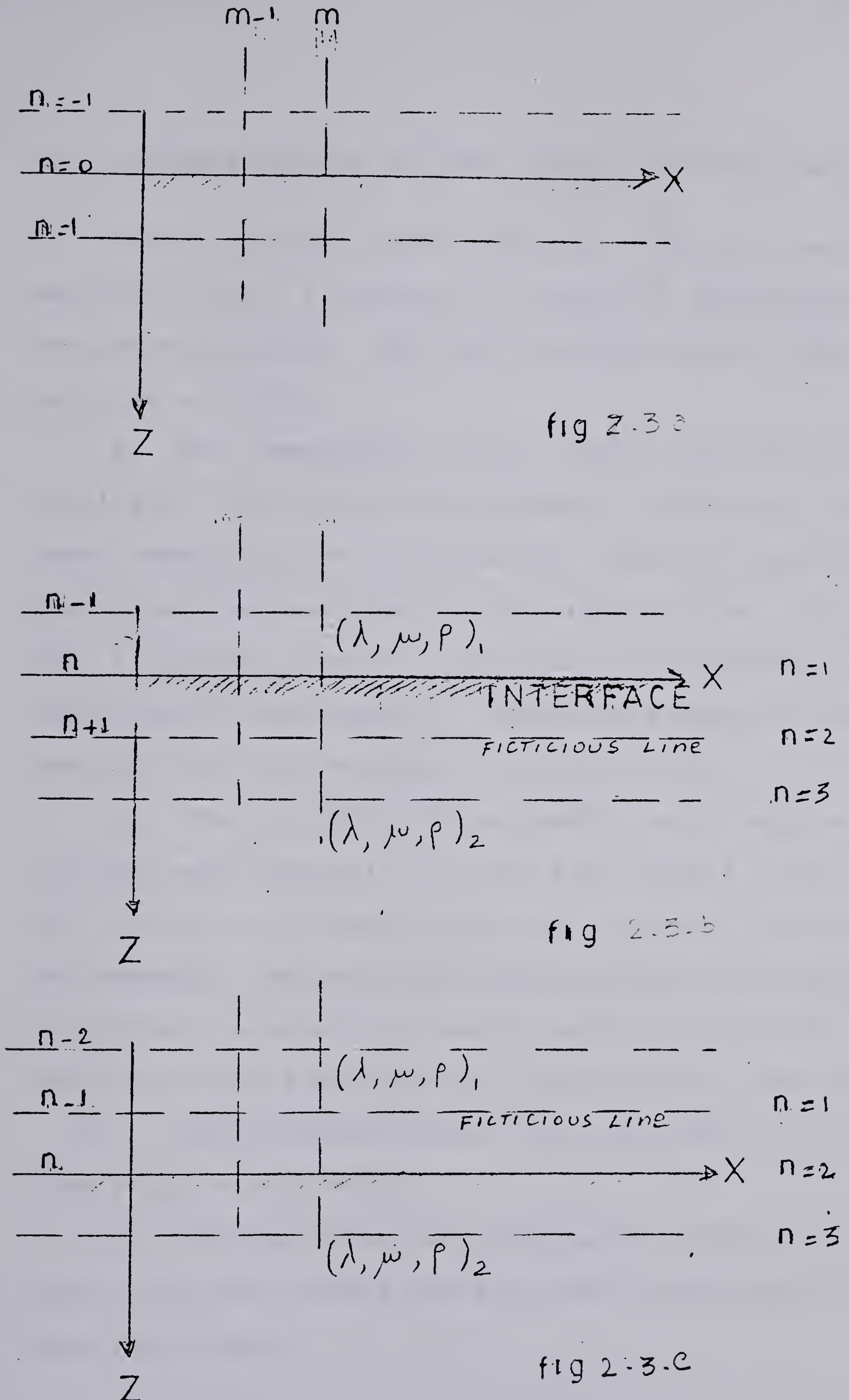


Fig. 2.3. The "homogeneous media" method

2.3.a Extension of the media by fictitious line at the surface

2.3.b The upper medium is extended at the interface

2.3.c Extension of the lower medium at the interface

2.6 PROBLEMS RELATED TO USUAL FINITE DIFFERENCE APPROACH

Since the usual finite difference approach has mainly an empirical basis, a plethora of "numerical experiments" have been published. The main problems are well defined by Kelly et al (1976):

i) The "homogeneous media" method, although taking explicitly into account the boundary conditions, gives undue instability at the interface. Numerous experiments and tests have been done on this effect (Ilian, 1975). This is generally due to a poor method of satisfying the conditions at the boundary. Besides the method is cumbersome and can only be applied to simple cases.

ii) The so called "heterogeneous media" approach, although more versatile, has the same problem as the above for the free interface and has poor record of stability at any boundary. Besides these major problems the question of satisfying the equation boundary conditions has been avoided and the approximations and errors are ignored.

iii) The two above methods give discrepant results (see Kelly et al, 1976).

iv) The connection between those developments and the other numerical methods based on other principles has not been established.

v) The treatment of the source is cumbersome and its numerical evaluation can be simplified.

Note: Generally the validity of variational principle or energy methods are established in the static case.

Since there is no restriction on the forces we can treat the mass × acceleration term of the equations of motion as an additional body force (or nodal force).

In this case, the problem is equivalent to considering the displacement continuous and linear between two time samples. If F is such a force the work during an interval of time t-tΔ/2 and t+Δt/2 :

$$\int_{t-\Delta t/2}^{t+\Delta t/2} F\cdot(U^{\frac{1}{2}}-U^{-\frac{1}{2}})\,dt = F_{mean}\cdot$$

$$= F\cdot(U^{\frac{1}{2}}-U^{-\frac{1}{2}})$$

This work being equal to the variation of kinetic energy, we have:

$$(F,(U^{\frac{1}{2}}-U^{-\frac{1}{2}})) = \frac{m}{2}\cdot((\dot{U}^{\frac{1}{2}})^2 - (\dot{U}^{-\frac{1}{2}})^2)$$

$$\text{i.e. } m\ddot{u}\cdot(U^{\frac{1}{2}}-U^{-\frac{1}{2}}) = (AU,\ (U^{\frac{1}{2}}-U^{-\frac{1}{2}}))$$

which implies

$$\rho\ddot{u} = F_{mean}.$$

CHAPTER III

THE PROPOSED NUMERICAL SOLUTION

The aim of this chapter is to develop a second order finite difference scheme valid in anisotropic media and complicated structure.

To do so, we will have first to consider a basic topological question: The parameter distributions at the boundary and their specificity with regard to a given operator.

3.1.a Spatial Discretization and Operator Decomposition

Let us consider at a given instant $\Gamma = \ell\Delta t$ a point $P\ (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^n$ with

$$x_1 = mh_1$$

$$x_2 = rh_2$$

$$x_3 = nh_3$$

An elementary region such that $P \in \Omega_h$ can be defined as $P\ (x_1 \pm h_1, x_2 \pm h_2, x_3 \pm h_3)$ and since we operate in a discrete, finite space the point P will have a finite number of points in its neighborhood. i.e. if the number of spatial dimensions is 2 we have:

$$nbh(P_0) = \{P_1, \ldots, P_8\} \subset \mathbb{D}(A) = \Omega_h$$

where $\mathbb{D}(A)$ represents the elementary domain of the operator A (fig. 3.1).

$\mathbb{D}$ can also be decomposed into elemental subdomains $\mathbb{D}^{\dot{I}}$ such that

$$\mathbb{D} = \bigcup_{I=1}^{2^n} \mathbb{D}^{\dot{I}}$$

where $\dot{I} = \mathrm{I, II, III, IV}$ if $n = 2$

$I = \mathrm{I}, \ldots, \mathrm{VIII}$ if $n = 3$

spatial dimensions

Since the operator A can be decomposed in differential operators of the form $\{a_{ij} D_i D_j\}^{\dot{I}}$ where $\{a_{ij}\}^{\dot{I}}$ is the parameter corresponding to the elemental subdomain $\mathbb{D}^{\dot{I}}$ we can write

$$\sum_{I=1}^{2^n} (a_{ij} D_i D_j)^{\dot{I}} U = \sum_{I=1}^{2^n} (D_i a_{ij} D_j)^{\dot{I}} U$$

Consequently, in a discrete space the "homogeneous media" and "heterogeneous media" formulations should be numerically and algorithmically identical. We will come back later to this problem.

3.1.b Specific Distribution of the Parameters

Let us consider the application of a differential operator, D_x for instance, onto the previously defined domain.

If $\mathbb{D}(D_x)$ is the domain of the operator D_x, we can write:

$$\mathbb{D}(D_x^I U) = \{U(P_1),\ U(P_0)\} \subset \mathbb{D}^I \qquad (3.1.1)$$

and $\mathbb{D}(D_x^I U)$ is the restriction of the operator domain.

The property (3.1.1) remains valid if we multiply U by a coefficient λ, then

$$\mathbb{D}(D_x^I \lambda U) = \mathbb{D}(D_x^I (U)) \subset \mathbb{D}^I \qquad (3.1.2)$$

which implies that

$$\mathbb{D}(\lambda)^I = \mathbb{D}(D_x^I (U)) \subset \mathbb{D}^I \qquad (3.1.3)$$

Hence, the parameter value can be restricted to the differential operator domain.

Since we have by definition

$$\text{Measure } (h_1) = |U(P_1) - U(P_0)| \qquad \text{with } h_1 = (P_0, P_1)$$

(Courant and Hilbert, 1961)

h_1 being a Lebesgue Measure if $U(P_0) = m\ h_1 = x$ and $UP_1 = (m + 1)h_1$.

Hence, we can write

$$D_x^I \lambda U = \frac{\lambda^I}{h_1} \text{ Measure } (P_0, P_1) \tag{3.1.4}$$

Then, the relative change of the grid size has the same effect as a variation of parameters.

These remarks are consistent with the fact that in elasticity the factors $\bar{\lambda}\Delta t^2/\rho h^2$, $\mu\Delta t^2/\rho h^2$ are dimensionless. Consequently, the values of the parameters do not have to be unique in each elemental subspace $\mathbb{D}^I$, and can have different values corresponding to the operator domain restriction.

Hence, the boundaries are not limited to the basic rectangular grid shape and can be diagonal, which allows more precise contours.

So, eight parameter values can be allowed in the neighborhood of P (fig.3.1).

If we call

$$\lambda_1 = (\lambda_1^+ + \lambda_1^-)/2$$

$$\lambda_2 = (\lambda_2^+ + \lambda_2^-)/2$$

$$\lambda_3 = (\lambda_3^+ + \lambda_3^-)/2$$

$$\lambda_4 = (\lambda_4^+ + \lambda_4^-)/2 \tag{3.1.5}$$

The coefficients relevant to the operator D_x, for instance, will be λ_1 and λ_3 and λ_2, λ_4 will concern the operator D_z. In the case of mixed derivatives, D_xD_z, for instance, we remark that the system can always be written in a conservation law form (Lax and Wendroff, 1960)

$$D_t(D\underset{\sim}{U}_t) = D_x(\sigma_{xx}) + D_z(\sigma_{xz})$$

or

$$D_t(D\underset{\sim}{U}_t) = D_z(\sigma_{zz}) + D_x(\sigma_{zx}) \qquad (3.1.6)$$

Consequently, the parameters are relevant to the domain of the second derivative.

These remarks are particularly important since they establish the relationship between

- Parameters and Grid
- Parameters and Operator.

We note considering the domain D^I for instance

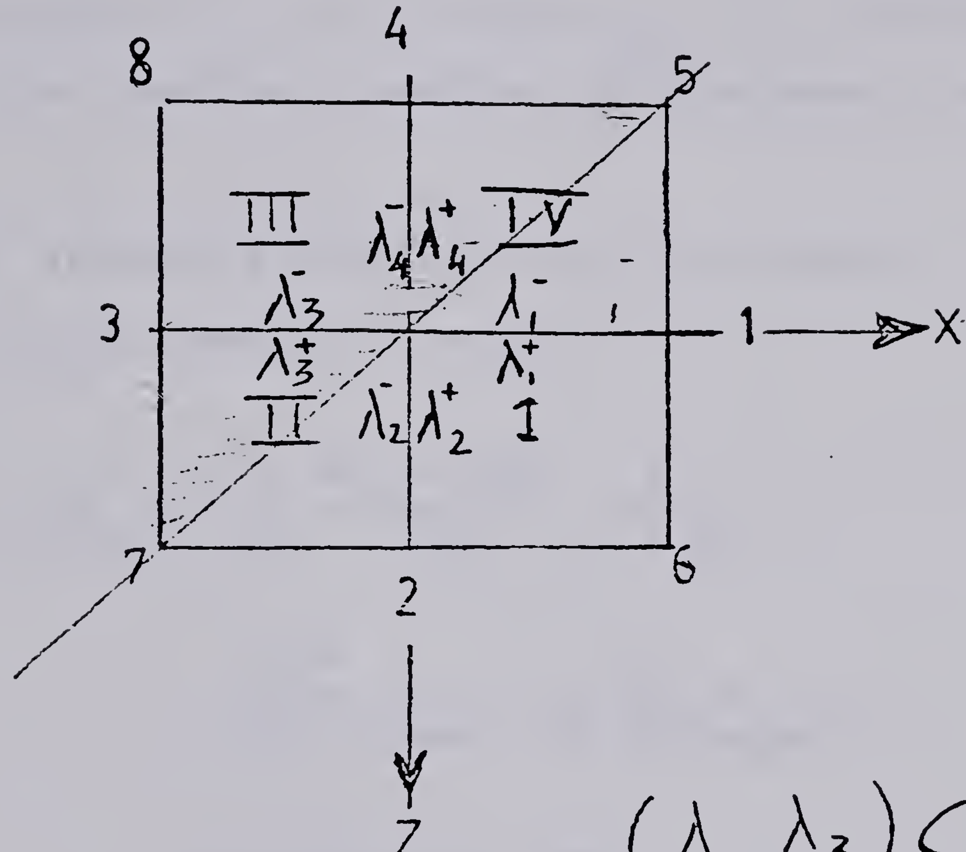
$$\mathbb{D}^I(D_{xx}U) \Rightarrow u \in [0\ x) \cap \mathbb{D}^I = \{u_0, u_1\}$$

$$\mathbb{D}^I(D_{zz}U) \Rightarrow u \in [0\ z) \cap \mathbb{D}^I = \{u_0, u_2\}$$

$$\mathbb{D}^I(D_xU_z) \Rightarrow u \in ([0x) \times [0z)) \cap \mathbb{D}^I$$

$$= \{u_0, u_1, u_2, u_6\} \qquad (3.1.7)$$

Fig. 3.1. Distribution of the parameters at the boundary



$$(\lambda_1, \lambda_3) \subset \mathbb{D}(D_x)$$

$$(\lambda_2, \lambda_4) \subset \mathbb{D}(D_z)$$

$$(\lambda_1^+, \lambda_2^+) \subset \mathbb{D}^{I}$$

$$\lambda_1^+ \subset \bar{\mathbb{D}}(D_x^{I})$$

$$\lambda_2^+ \subset \mathbb{D}(D_z^{I})$$

$$\lambda_2^- \subset \mathbb{D}(D_z^{II})$$

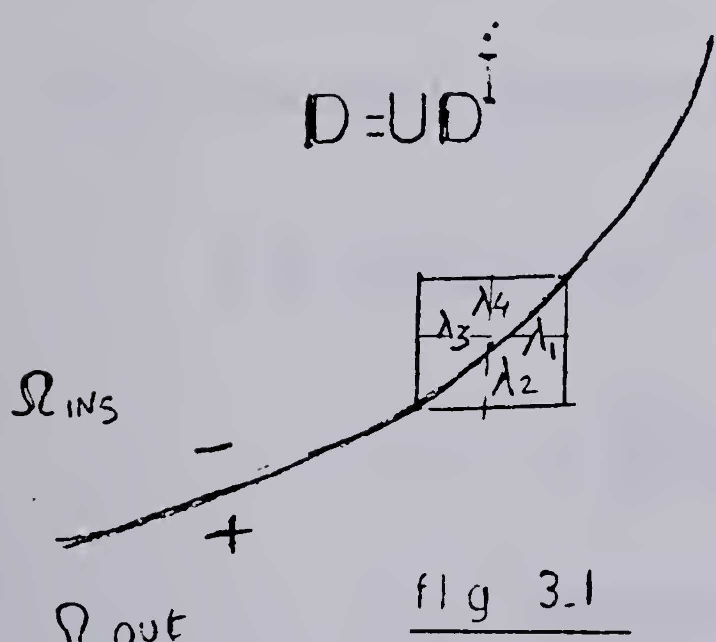


fig 3.1

3.2 TAYLOR DEVELOPMENT OF THE DERIVATIVES

Instead of developing the differential operators independently of the parameters we shall develop the terms $a_{ij}D_iD_jU$ in each subdomain $\mathbb{D}^{i} \subset \mathbb{D}$.

Let us suppose for convenience $a_{11} = \lambda/\rho$ which is the case of the equation of motion (displacement) in a liquid.

We have, according to fig. 3,2in the subspace $\mathbb{D}^{I} = \mathbb{D}^{1}$ at a given instant $t = P\Delta t$

$$\frac{1}{2}\,\frac{\lambda}{\rho}\,\rho_1 h_1 D^2_{xx}U = \lambda_1^+ \frac{[U_1 - U_0]}{h_1} - \lambda_1^+ D_x U$$

$$- \frac{\lambda_1^+ h_1^2}{6} D^3_{xxx}U - \lambda_1^+ \frac{h_1^3}{24} D^4_{xxxx}U + \ldots$$

$$\text{with } \mathbb{D}(D_{xx}U) \subset \mathbb{D}^{I} \qquad (3.2.1)$$

In the elemental subspace $\mathbb{D}^{i} = \mathbb{D}^{II}$ we can write

$$\frac{1}{2}\,\frac{\lambda}{\rho}\,\rho_2 h_3 D^2_{xx}U = \lambda_3^+ \frac{[U_3 - U_0]}{h_3} + \lambda_2^+ D_x U$$

$$+ \frac{\lambda_3^+ h_3^2}{6} D^3_{xxx}U - \frac{\lambda_3^+ h_3^3}{24} D^4_{xxxx}U + \ldots$$

$$\text{with } \mathbb{D}(\lambda/\rho D_{xx}) \subset \mathbb{D}^{II} \qquad (3.2.2)$$

After obtaining similar developments in the subspace $\mathbb{D}^{III}$ and $\mathbb{D}^{IV}$, we obtain after summation (3.2.2)

$$\frac{\lambda}{\rho} D_{xx}U = \frac{2}{h_1\rho_1 + h_3\rho_3} \{\lambda_1[U_1 - U_0]/h_1 + \lambda_3[U_3 - U_0]/h_3$$

$$+ [\lambda_3 D_x U - \lambda_1 D_x U] + [\lambda_3 h_3^2 - \lambda_1 h_1^2]D_{xxx}U$$

$$- \frac{1}{24} [\lambda_1 h_1^3 + \lambda_3 h_3^3]D_{xxxx}U + \ldots \qquad (3.2.3)$$

with $\mathbb{D}(\lambda/\rho D_{xx}) \subset \mathbb{D}$

and $\lambda_1 = (\lambda_1^+ + \lambda_1^-)/2 \quad , \quad \rho_1 = (\rho_1^+ + \rho_1^-)/2$

$\lambda_3 = (\lambda_3^+ + \lambda_3^-)/2 \quad , \quad \rho_3 = (\rho_3^+ + \rho_3^-)/2$

by the same method, we obtain in $\mathbb{D}(\lambda/\rho D_{zz}) \subset \mathbb{D}$:

$$\frac{\lambda}{\rho} D_{zz}U = \frac{2}{\rho_2 h_2 + \rho_2 h_4} \{\lambda_2/h_2\ [U_2 - U_0] + \lambda_4/h_4[U_4 - U_0]$$

$$+ [\lambda_4 D_2 U - \lambda_2 D_z U] + [\lambda_4 h_4^2 - \lambda_2 h_2^2]D_{zzz}U$$

$$- \frac{1}{24} [\lambda_2 h_1^3 + \lambda_4 h_3^3]D_{xxxx}U + \ldots \qquad (3.2.4)$$

where $\lambda_2 = (\lambda_2^+ + \lambda_2^-)/2 \quad , \quad \rho_2 = (\rho_2^+ + \rho_2^-)/2$

$\lambda_4 = (\lambda_4^+ + \lambda_4^-)/2 \quad , \quad \rho_4 = (\rho_4^+ + \rho_4^-)/2$

The expressions (3.2.3) and (3.2.4) yield, Δt being the time increment:

$$\frac{\lambda}{\rho} \Delta t^2 D_{xx} U = \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3} \{[\lambda_1/h_1[U_1 - U_0] + \lambda_3/h_3[U_3 - U_0]]$$

$$+ [\lambda_3 D_x U - \lambda_1 D_x U]\} + \frac{\Delta t^2}{h^2} (\lambda_3 - \lambda_1) O(h^3 D_{xxx} U)$$

$$- \frac{t^2}{h^2} O\ (\lambda h^4 D_{xxxx} U) \tag{3.2.5}$$

$$\frac{\lambda}{\rho} \Delta t^2 D_{zz} U = \frac{2\Delta t^2}{h_2\rho_2 + h_2\rho_4} \{[\lambda_2/h_2[U_2 - U_0] + \lambda_4/h_2[U_4 - U_0]]$$

$$+ [\lambda_4 D_2 U - \lambda_2 D_2 U]\} + \frac{\Delta t^2}{h^2} O\ [(\lambda_4 - \lambda_1) h^3 D_{zzz} U]$$

$$+ \frac{\Delta t^2}{h^2} O\ (\lambda h^4 D_{zzzz} U) \tag{3.2.6}$$

Since $\lambda\Delta t^2/\rho$ has in elasticity the dimensions of a $(\text{length})^2$ the discretisation error of the above expressions will be of order

$O(h^3)_\Gamma$ if P is a boundary point

$O(h^4)$ if P is an inner point (or if the boundary is parallel to the domain of differentiation)

The expressions $[\lambda_3 D_x U - \lambda_1 D_x U]$ and $[\lambda_4 D_z U - \lambda_2 D_z U]$ are boundary values naturally introduced in the equations; we will keep them in their classical differential form, and take into account the boundary conditions only after replacement of the second order differential terms of the equations of motion by the above developments.

We note that for SH waves or a liquid medium the abovementioned boundary terms cancel. This shows that the "heterogeneous media" development given by Boore (1972) in the SH wave case, is correct, but this is not the case for the elastic motion as we shall see later.

3.2.b Development of Expressions of the Form $\lambda/\rho\ D_i D_j U$

By Taylor Series we have for $D^I(D_{xz}) \subset D^I$

$$U_{6\,(\varepsilon D^t)} = U_0 + h_1 D_x U + h_1^2/2\ D_{xx}^2 U + h_1 h_2 D_{xz} U$$
$$+ h_2^2/2\ D_z D_z U + \frac{1}{3!} (D_x U + D_z U)^{(3)}$$
$$+ \frac{1}{4!} (D_x U + D_z U)^{(4)} + \ldots \qquad (3.2.7)$$

Replacing in the above expression U_1, U_2 by their developments we obtain:

$$U_6 = U_1 + U_2 - U_0 + h_1 h_2 D_{xz} U$$

$$+ \frac{1}{6} [3h_1 h_2^2 D_{xzz} U + 3h_1^2 h_2 D_{xxz} U] + 0(h^4) \qquad (3.2.8)$$

which leads to

$$\Delta t^2 \lambda/\rho \; h_1 \rho_1^+ D_{zx} U = \Delta t^2 \frac{\lambda_1^+}{h_2} \{[(U_6 - U_1) - (U_2 - U_0)]$$

$$+ 0(h^3)_\Gamma + 0(h^4)\}(U \varepsilon D^I) \qquad (3.2.9)$$

$$\Delta t^2 \lambda/\rho \; h_1 \rho_1^- D_{zx} U = \Delta t^2 \frac{\lambda_1^-}{h_4} \{[(U_1 - U_5) - (U_0 - U_4)]$$

$$+ 0(h^3)_\Gamma + 0(h^4)\}(U \varepsilon D^{IV}) \qquad (3.2.10)$$

$$\Delta t^2 \lambda/\rho \; h_3 \rho_3^+ D_{zx} U = \Delta t^2 \frac{\lambda_3^+}{h_2} \{[(U_7 - U_3) - (U_2 - U_0)]$$

$$- 0(h^3)_\Gamma + 0(h^4)(U \varepsilon D^{II}) \qquad (3.2.11)$$

$$\Delta t^2 \lambda/\rho \; h_3 \rho_3^+ D_{zx} U = \Delta t^2 \frac{\lambda_3^-}{h_2} \{[(U_0 - U_4) - (U_3 - U_8)]$$

$$- 0(h^3)_\Gamma + 0(h^4)(U \varepsilon D^{III}) \qquad (3.2.12)$$

Summation of equations (3.2.9) to (3.2.12) yield:

$$\Delta t^2 \frac{\lambda}{\rho} D_{zx}U = \frac{\Delta t^2}{2(h_1\rho_1 + h_1\rho_{3})} \{\lambda_1^+/h_2[(U_6 - U_1) - (U_2 - U_0)]$$

$$+ [\lambda_1^-/h_4 \ (U_1 - U_5) - (U_0 - U_4)]$$

$$+ \lambda_3^+/h_2[(U_2 - U_0) - (U_7 - U_3)]$$

$$+ \lambda_3^-/h_4[(U_0 - U_4) - (U_3 - U_8)]\}$$

$$+ 0(h^3)_\Gamma + 0(h^4) \qquad (3.2.13)$$

The term $0(h^3)$ is of the form $\frac{(\lambda_1 - \lambda_3)}{h^3} \Delta t^2 \ 0(h^3)$. This term cancels if U_0 is an inner point. In this case the development is of order $0(h^4)$.

By the same way

$$\Delta t^2 \frac{\lambda}{\rho} D_{xz}U = \frac{\Delta t^2}{2(h_2\rho_2 + h_4\rho_{4})} \{\lambda_2^+/h_1[(U_6 - U_2) - (U_1 - U_0)]$$

$$+ \lambda_2^-/h_3[(U_2 - U_7) - (U_0 - U_3)]$$

$$+ \lambda_4^-/h_3[(U_0 - U_3) - (U_4 - U_8)]$$

$$+ \lambda_4^+/h_1[(U_1 - U_0) - (U_5 - U_4)]\} \qquad (3.2.14)$$

If U_0 is an inner point (3.2.13) and (3.2.14) reduced to

$$\Delta t^2 \frac{\lambda}{\rho} D_{xz}U = \Delta t^2 \frac{\lambda}{\rho} D_{xz}U$$

with

$$\Delta t^2 \frac{\lambda}{\rho} D_{xz}U = \Delta t^2 \frac{\lambda}{\rho(h_2 + h_4)}[(U_6 + U_8)$$

$$- (U_5 + U_7)]/(h_1 + h_3) + 0(h^4) \qquad (3.2.15)$$

To underline the boundary conditions (3.2.13), (3.2.14) can be written:

$$\Delta t^2 \frac{\lambda}{\rho} D_{zx}U = \frac{\Delta t^2}{2(h_1\rho_1 + h_3\rho_3)} \{\lambda_1^+(U_6 - U_1)/h_2$$

$$+\lambda_1^-(U_1 - U_5)/h_4 - \lambda_3^+(U_7 - U_3)/h_2$$

$$- \lambda_3^-(U_3 - U_8)/h_4 -(\lambda_1^+D_z^+ - \lambda_1^-D_z^-)U$$

$$+(\lambda_3^+D_z^+ + \lambda_3^-D_z^-)U\}$$

$$+ \frac{\Delta t}{h^2} (\lambda_1 - \lambda_3)0(h^2D_{zz}U) + 0(h^4) \qquad (3.2.16)$$

$$\Delta t^2 \frac{\lambda}{\rho} D_{xz}U = \frac{\Delta t^2}{2(h_2\rho_2 + h_4\rho_4)} \{\lambda_2^+(U_6 - U_2)/h_1$$

$$+\lambda_2^-(U_2 - U_7)/h_3 - \lambda_4^-(U_4 - U_8)/h_3$$

$$-\lambda_4^+(U_5 - U_4)/h_1 - \lambda_2^+ D_x^+ U - \lambda_2^- D_x^-$$

$$+\lambda_4^+ D_x^+ U + \lambda_4^- D_x^- \}$$

$$+ \frac{\Delta t^2}{h^2} (\lambda_4 - \lambda_2) 0(h^2 D_{xx}U) + 0h^4 \qquad (3.2.17)$$

the expressions (3.2.16), (3.2.17) being equivalent to (3.2.16), (3.2.17).

For convenience if FD(.) represents the terms of the finite difference development independent of the derivatives; the equations (3.2.5), (3.2.6), (3.2.13) and (3.2.14) yield

$$\frac{\lambda}{\rho} \Delta t^2 D_{xx}U = F.D(\frac{\lambda}{\rho} \Delta t^2 D_{xx}^2 U)$$

$$+ \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\lambda_3 D_x^- U - \lambda_1 D_x^+ U]$$

$$+ 0(h^3)_\Gamma + 0(h^4) \qquad (3.2.18)$$

$$\frac{\lambda}{\rho} \Delta t^2 D_{zz} U = F.D(\frac{\lambda}{\rho} \Delta t^2 D_{zz} U)$$

$$+ \frac{2\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\lambda_4 D_z^- U - \lambda_2 D_2^+ U]$$

$$+ O(h^3)_\Gamma + O(h^4) \tag{3.2.19}$$

$$\frac{\lambda}{\rho} \Delta t^2 D_{xz} U = F.D(\frac{\lambda}{\rho} \Delta t^2 D_{xz} U)$$

$$- \frac{\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\lambda_2 D_x - \lambda_4 D_x] U$$

$$+ O(h^2)_\Gamma + O(h^4) \tag{3.2.20}$$

$$\frac{\lambda}{\rho} \Delta t^2 D_{zx} U = F.D(\frac{\lambda}{\rho} \Delta t^2 D_{zx} U)$$

$$- \frac{\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\lambda_1 D_z - \lambda_3 D_z] U$$

$$+ O(h^2)_\Gamma + O(h^4) \tag{3.2.21}$$

since consequently to section 3.1

$$\lambda_1 D_z = (\lambda_1^- D_z^- + \lambda_1^+ D_z^+)/2$$

$$\lambda_3 D_z = (\lambda_3^+ D_z^+ + \lambda_3^- D_z^-)/2 \text{ etc.} \tag{3.2.22}$$

3.3 FINITE DIFFERENCE DEVELOPMENT OF THE EQUATIONS OF MOTION (DISPLACEMENT VECTOR) WITH ARBITRARY BOUNDARIES

From (2.1.12) we have in the (x,z) plane:

$$D_{tt}U = (\lambda + 2\mu)/\rho D_{xx}U + \frac{\lambda}{\rho} D_{zx}W + \frac{\mu}{\rho}(D_{xz}W + D_{zz}U) \quad (3.3.1)$$

$$D_{tt}W = (\lambda + 2\mu)/\rho D_{zz}W + \frac{\lambda}{\rho} D_{xz}U + \frac{\mu}{\rho}(D_{zx}U + D_{xx}W) \quad (3.3.2)$$

The auxilliary conditions being given by

$$U^+ = U^-$$

$$W^+ = W^- \quad (3.3.3)$$

$$[(\lambda + 2\mu)D_xU + \lambda D_zW]^+ = [(\lambda + 2\mu)D_xU + \lambda D_zW]^- \quad (3.3.4)$$
(vertical boundary)

$$[(\lambda + 2\mu)D_zW + \lambda D_xU]^+ = [(\lambda + 2\mu)D_zW + \lambda D_xU]^- \quad (3.3.5)$$
(horizontal boundary)

$$(\mu D_xW + \mu D_zU)^+ = (\mu D_xW + \mu D_zU)^- \quad (3.3.6)$$

where the sign + designs the exterior part of the boundary.

We note that condition (3.3.3) expresses the displacement continuity, the normal and tangential stress continuity being respectively expressed by conditions (3.3.4), (3.3.5) and (3.3.6). We note:

Vertical boundary conditions are expressed by (3.3.4), (3.3.6)

Horizontal boundary conditions are expressed by (3.3.5), (3.3.6).

The free boundary case (i.e. surface condition) is expressed by

$$\left.\begin{aligned}\sigma_{zx} &= 0\\ \sigma_{xx} &= 0\end{aligned}\right\} \text{(vertical free boundary)}$$

$$\left.\begin{aligned}\sigma_{zz} &= 0\\ \sigma_{xz} &= 0\end{aligned}\right\} \begin{array}{l}\text{(horizontal free boundary)}\\ \text{(with } \sigma_{zx} = \sigma_{xz})\end{array}$$

i.e.

$$[(\lambda + 2\mu)D_x U + \lambda D_z W] = 0 \qquad (3.3.7)$$

$$[(\lambda + 2\mu)D_z W + \lambda D_x U] = 0 \qquad (3.3.8)$$

$$\mu D_x W + \mu D_z U = 0 \qquad (3.3.9)$$

The initial conditions and the source problem will be considered in the next section.

Replacing relations (3.2.18) to (3.2.21) into equations (3.3.1) and (3.3.2) yield for the general case:

$$F.D(D_{tt}U) = F.D\{(\lambda + 2\mu)/\rho D_{xx}U + \frac{\lambda}{\rho} D_{zx}W$$

$$+ \mu[D_{xz}W + D_{zz}U]\}$$

$$+ \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3} [(\lambda + 2\mu)_3 D_x^- U - [\lambda + 2\mu]_1 D_x^+ U]$$

$$+ \frac{2\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\mu_4 D_z^- W - \mu_2 D_z^+ U]$$

$$+ \frac{\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\lambda_3 D_z W - \lambda_1 D_z W]$$

$$+ \frac{\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\mu_4 D_x W - \mu_2 D_x W]$$

$$+ 0(h^2)_\Gamma + 0(h^4) \qquad (3.3.10)$$

$$F.D(D_{tt}W) = F.D\{(\lambda + 2\mu)/\rho D_{zz}W + \frac{\lambda}{\rho} D_{xz}U$$

$$+ \mu[D_{zx}U + D_{xx}W]\}$$

$$+ \frac{2\Delta t^2}{h_2\rho_2 + h_4\rho_4} [(\lambda + 2\mu)_4 D_z^- W - (\lambda + 2\mu)_2 D_z^+ W]$$

$$+ \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\mu_3 D_x^- W - \mu_1 D_x^+ W]$$

$$+ \frac{\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\lambda_4 D_x U - \lambda_2 D_x U]$$

$$+ \frac{\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\mu_3 D_z U - \mu_1 D_z U]$$

$$+ 0(h^2)_\Gamma + 0(h^4) \qquad (3.3.11)$$

$$F.D(D_{tt}V) = F.D\{\frac{\mu}{\rho} D_{zz}V + \frac{\mu}{\rho} D_{zz}V\}$$

$$+ \frac{2\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\mu_4 D_z^- V - \mu_2 D_z^+ V]$$

$$+ \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\mu_3 D_x^- V - \mu_1 D_x^+ V]$$

$$+ 0(h^3)_\Gamma + 0(h^4) \qquad (3.3.12)$$

Substituting the boundary conditions (3.3.4), (3.3.5) and (3.3.6) into (3.3.10), (3.3.11) and (3.3.12) yield

$$F.D(D_{tt}U) = F.D\{(\lambda + 2\mu)/\rho D_{xx}U + \frac{\lambda}{\rho} D_{zx}W$$

$$+ \mu[D_{xz} + D_{zz}U]\}$$

$$+ \frac{\Delta t^2}{h_1\rho_1 + h_3\rho_3} [\lambda_1 D_z W - \lambda_3 D_z W]$$

$$+ \frac{\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\mu_4 D_x W - \mu_2 D_x W]$$

$$+ 0(h^2)_\Gamma + 0(h^4) \qquad (3.3.13)$$

$$F.D.(D_{tt}W) = F.D\{(\lambda + 2\mu)/\rho D_{zz}W + \frac{\lambda}{\rho} D_{xz}$$

$$+ \mu[D_{zx} + D_{xx}W]\}$$

$$+ \frac{\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\lambda_2 D_x U - \lambda_4 D_x U]$$

$$+ \frac{\Delta t^2}{h_3\rho_3 + h_1\rho_1} [\mu_1 D_z U - \mu_3 D_z U]$$

$$+ 0(h^2)_\Gamma + 0(h^4) \qquad (3.3.14)$$

$$F\ D(D_{tt}V) = F\ D\{\frac{\mu}{\rho} (D_{zz}V + D_{xx}V)\}$$

$$+ 0(h^3)_\Gamma + 0(h^4) \qquad (3.3.15)$$

3.4 EXPLICIT SCHEME

After substituting the F D{•} values given by relations (3.2.18) to (3.2.20) into equations (3.3.13) to (3.3.15), and replacing the first derivatives by their second order development we obtain:

$$U^1 = 2U-U^{-1} + \frac{2\Delta t^2}{h_1\rho_1+h_3\rho_3}\{-\frac{(\lambda_1+2\mu_1)}{h_1}[U_1-U_0] + \frac{(\lambda_3+2\mu_3)}{h_3}(U_3-U_0)\}$$

$$+ \frac{\Delta t^2}{2(h_1\rho_1+h_3\rho_3)} \{\frac{\lambda_1^+}{h_2} [(W_6-W_1) + (W_8-W_3)] + \frac{\lambda_1^-}{h_4}[(W_1-W_5)$$

$$+ (W_0-W_4)] + \frac{\lambda_3^+}{h_2} [(W_3-W_7) + (W_0-W_2)]$$

$$- \frac{\lambda_3^-}{h_4} [[W_8-W_3) + (W_4-W_0)]\}$$

$$+ \frac{2\Delta t^2}{h_2\rho_2+h_4\rho_4}\left\{\frac{\mu_2}{h_2}[U_2-U_0] + \frac{\mu_4}{h_4}[U_4-U_0]\right\}$$

$$+ \frac{\Delta t^2}{2(h_2\rho_2+h_4\rho_4)}\left\{\left[\frac{\mu_2^+}{h_1}[(W_6-W_2) + (W_1-W_0)]\right.\right.$$

$$+ \frac{\mu_2^-}{h_3}[(W_2-W_7) + (W_0-W_3)] + \frac{\mu_4^-}{h_3}[(W_8-W_4)$$

$$+ (W_3-W_0)] + \frac{\mu_4^+}{h_1}[(W_4-W_5) + (W_0-W_1)]\}$$

$$+ O(h^2)_\Gamma + O(h^4) \qquad (3.4.1)$$

$$W^1 = 2W-W^{-1} + \frac{2\Delta t^2}{h_2\rho_2+h_4\rho_4}\left\{\frac{(\lambda_2+2\mu_2)}{h_2}(W_2-W_0)\right.$$

$$+ \frac{(\lambda_4+2\mu_4)}{h_4}(W_4-W_0)\} + \frac{\Delta t^2}{2(h_2\rho_2+h_4\rho_4)}$$

$$\left\{\frac{\lambda_2^+}{h_1}[(U_6-U_2) + (U_1-U_0)]\right.$$

$$+ \frac{\lambda_2^-}{h_3}[(U_2-U_7) + (U_0-U_3)]$$

$$+ \frac{\lambda_4^-}{h_3}[(U_8-U_4) + (U_3-U_0)]$$

$$+ \frac{\lambda_4^+}{h_1}[(U_4-U_5) + (U_0-U_1)]$$

$$+ \frac{2\Delta t^2}{h_1\rho_1+h_3\rho_3}\left\{\frac{\mu_1}{h_1}(W_1-W_0) + \frac{\mu_3}{h_3}(W_3-W_0)\right\}$$

$$+ \frac{\Delta t^2}{2(h_1\rho_1+h_3\rho_3)}\left\{\frac{\mu_1^+}{h_2}[(U_6-U_1) + (U_2-U_0)]\right.$$

$$+ \frac{\mu_1^-}{h_4} [(U_1-U_5) + (U_0-U_4)] + \frac{\mu_3^-}{h_4} [(U_8-U_3)$$

$$+ (U_4-U_0)] + \frac{\omega_3^+}{h_2} [(U_3-U_7) + (U_0-U_2)]\}$$

$$+ O(h^2)_\Gamma + O(h^4) \qquad (3.4.2)$$

$$V^1 = 2V-V^{-1} + \frac{2\Delta t^2}{h_2\rho_2+h_4\rho_4} [\frac{\mu_2}{h_2} [V_2-V_0]$$

$$+ \frac{\omega_4}{h_4} [V_4-V_0]] + \frac{2\Delta t^2}{h_1\rho_1+h_3\rho_3} [\frac{\omega_1}{h_1} (V_1-V_0)$$

$$+ \frac{\omega_3}{h_3} (V_3-V_0)]$$

$$+ O(h^3)_\Gamma + O(h^4) \qquad (3.4.3)$$

where $U = U^0_{m,n}$ $\quad U^1 = U^1_{m,n}$

$U_1 = U_{m+1,n}$ $\quad U_2 = U_{m,n+1}$

$U_3 = U_{m-1,n}$ $\quad U_4 = U_{m,n-1}$

$U_5 = U_{n-1,m+1}$ $\quad U_6 = U_{m+1,m+1}$

$U_7 = U_{m-1,n+1}$ $\quad U_8 = U_{m-1,n-1}$

$\lambda_i = (\lambda_i^+ + \lambda_i^-)/2$

$\mu_i = (\mu_i^+ + \mu_i^-)/2 \qquad i = 1,\ 2$

We remark:

i) The boundary conditions have been incorporated to the equations with a well defined second order accuracy at the boundary.

ii) The parameter distributions are governed by rules seen in section 3.2 and lead to a unique development for a given accuracy.

iii) The system satisfies arbitrary boundaries including diagonal boundaries, as far as those boundaries pass by the grid points, and also take into consideration velocity and grid anisotropy.

iv) No need for cumbersome "fictitious lines" such as defined by the "homogeneous media" method (Alterman et al (1968, 1970); Boore (1970); Kelly et al (1976); even in the case of free boundary conditions.

3.5 STRESS FREE BOUNDARY CASE

The equations are governed by relations (3.3.7) to (3.3.9). To satisfy those conditions as well as the equations of motion, it is sufficient to set the relevant parameters to zero in equations (3.4.1) to (3.4.3). Although the problem is automatically solved by equations (3.4.1) to (3.4.3), we will still write the developments relevant to different cases.

This problem has been the source of numerous papers since it concerns Rayleigh waves. The equations are less complicated, but we are in the case where there is maximum error at the boundary (consequently maximum instability for inadequate systems).

Two cases have to be considered; the general case where the medium is non-homogeneous on Γ^+, and the most common case when the medium is homogeneous on Γ^+.

3.5.a The Medium is Non-Homogeneous on Γ^+

3.5.a.1 Horizontal stress free surface

The boundary conditions are expressed by equations (3.3.8) and (3.3.9).

For a point $P_{m,0}$ at the surface the values of (λ, μ, ρ) relevant to the closed domain $\bar{D}^{III} \cup \bar{D}^{IV}$ are

i.e.

$$(\lambda, \mu, \rho)_4^{\pm} = 0$$

$$(\lambda, \mu, \rho)_1^{-} = 0$$

$$(\lambda, \mu, \rho)_3^{-} = 0 \qquad (3.5.1)$$

where $(\lambda, \mu, \rho)_3^{-}$ means $\lambda_3^- = 0$, $\mu_3^- = 0$ ...

Equations (3.4.1) to (3.4.3) become

$$U^1 = 2U - U^{-1} + \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3}\{(\lambda_1 + 2\mu_1)/h_1[U_1 - U_0]$$

$$+ (\lambda_3 + 2\mu_3)/h_3(U_3 - U_0)\} + \frac{\Delta t^2}{h_1\rho_1^+ + h_3\rho_3^+}$$

$$\{\frac{\lambda_1^+}{h_2}[(W_6 - W_1) + (W_2 - W_0)]$$

$$+ \frac{\lambda_3^+}{h_2}[(W_3 - W_7) + (W_0 - W_2)]\}$$

$$+ \frac{2\Delta t^2}{h_2^2\rho_2}\mu_2(U_2 - U_0)$$

$$+ \frac{\Delta t^2}{2h_2\rho_2}\{\frac{\mu_2^+}{h_1}\{[(W_6 - W_2) + (W_1 - W_0)]$$

$$+ \frac{\mu_2^-}{h_3}[(W_2 - W_7) + (W_0 - W_3)]$$

$$+ O(h^2) \qquad (3.5.2)$$

$$W^1 = 2W - W^1 + \frac{2\Delta t^2}{h_2\rho_2}\{\frac{(\lambda_2 + 2\mu_2)}{h_2}(W_2 - W_0)\}$$

$$+ \frac{\Delta t^2}{2h_2\rho_2}\{\frac{\lambda_2^+}{h_1}[(U_6 - U_2) + (U_1 - U_0)]$$

$$+ \frac{\lambda_2^-}{h_3}[(U_2 - U_7) + (U_0 - U_3)]\}$$

$$+ \frac{2\Delta t^2}{h_1\rho_1 + h_3\rho_3}\{\frac{\omega_1}{h_1}(W_1 - W_0) + \frac{\mu_3}{h_3}(W_3 - W_0)\}$$

$$+ \frac{\Delta t^2}{2(h_1\rho_1 + h_3\rho_3)}\{\frac{\omega_1^+}{h_2}[(U_6 - U_1) + (U_2 - U_0)]$$

$$+ \frac{\mu_3^+}{h_2}[(U_3 - U_7) + (U_0 - U_2)]\} + O(h^2) \qquad (3.5.3)$$

$$V^1 = 2V - V^{-1} + \frac{2\Delta t^2}{h_2 \rho_2} \frac{\mu_2}{h_2} (V_2 - V_0) + O(h^3) \qquad (3.5.4)$$

where $(\lambda, \mu, \rho)_3 = \frac{1}{2} (\lambda, \mu, \rho)_3^+$

$$(\lambda, \mu, \rho)_1 = \frac{1}{2} (\lambda, \mu, \rho)_1^+$$

since $\lambda_1 = (\lambda_1^- + \lambda_1^+)/2$ etc.

and $(\lambda, \mu, \rho)_3^+ = (\lambda, \mu, \rho)_1^+ = (\lambda, \mu, \rho)_2$

if the medium is homogeneous on Γ^+.

If we set in (3.4.1) to (3.4.3) the conditions at the surface

$$(\rho, \lambda, \mu)_4 = (\rho, \lambda, \mu)_2$$

$$(\rho, \lambda, \mu)_3^- = (\rho, \lambda, \mu)_3^+$$

$$(\rho, \lambda, \mu)_1^- = (\rho, \lambda, \mu)_1^+ \quad \text{(media extension)}$$

as well as

$$(W_4, U_4) = (W_2, U_2) \quad \text{(image condition)}$$

we obtain identical results than (3.5.2) to (3.5.4).

3.5.a.2 Vertical stress free surface

The boundary conditions are expressed by equations (3.3.7) and (3.3.9).

The parameter values relevant to the closed domain $\overline{\mathbb{D}}^{II} \cup \overline{\mathbb{D}}^{III}$ are nulls.

i.e.

$$(\lambda,\ \mu,\ \rho)_3^{\pm} = 0$$

$$(\lambda,\ \mu,\ \rho)_4^{-} = 0$$

$$(\lambda,\ \mu,\ \rho)_2^{-} = 0 \tag{3.5.5}$$

Equations (3.4.1) to (3.4.3) become

$$\begin{aligned}
U^1 = 2U-U^{-1} &+ \frac{2\Delta t^2}{\rho_1 h_1}\ (\lambda_1+2\mu_1)/h_1\ [U_1-U_0] \\
&+ \frac{\Delta t^2}{2h_1\rho_1}\ \{\frac{\lambda_1^+}{h_2}\ [(W_6-W_1) + (W_2-W_0)] \\
&+ \frac{\lambda_1^-}{h_4}\ [(W_1-W_5) + (W_0-W_4)]\} \\
&+ \frac{2\Delta t^2}{h_2\rho_2+h_4\rho_4}\ \{\frac{\mu_2}{h_2}\ [U_2-U_0] + \frac{\mu_4}{h_4}\ [U_4-U_0]\} \\
&+ \frac{\Delta t^2}{2(h_2\rho_2+h_4\rho_4)}\ \{\frac{\mu_2^+}{h_1}\ [(W_6-W_2) + (W_1-W_0)] \\
&+ \frac{\mu_4^+}{h_1}\ [(W_4-W_5) + (W_0-W_1)] + 0(h^2)
\end{aligned} \tag{3.5.6}$$

$$W^1 = 2W - W^{-1} + \frac{2\Delta t^2}{h_2\rho_2 + h_4\rho_4} \{ \frac{(\lambda_2 + 2\mu_2)}{h_2} [W_2 - W_0]$$

$$+ \frac{(\lambda_4 + 2\mu_4)}{h_4} [W_4 - W_0]\}$$

$$+ \frac{\Delta t^2}{2(h_2\rho_2 + h_4\rho_4)} \{[\frac{\lambda_2^+}{h_1} [(U_6 - U_2) + (U_1 - U_0)]$$

$$+ \frac{\lambda_4^+}{h_1} [(U_4 - U_5) + (U_0 - U_1)]\}$$

$$+ \frac{2\Delta t^2}{h_1\rho_1} \frac{\mu_1}{h_1} [W_1 - W_0]$$

$$+ \frac{\Delta t^2}{2h_1\rho_1} \{ \frac{\mu_1^+}{h_2} [(U_6 - U_1) + (U_2 - U_0)]$$

$$+ \frac{\mu_1^-}{h_4} [(U_1 - U_5) + (U_0 - U_4)]\} + O(h^2) \qquad (3.5.7)$$

$$V^1 = 2V - V^{-1} + \frac{2\Delta t^2}{h_2\rho_2 + h_4\rho_4} [\frac{\mu_2}{h_2} [V_2 - V_0]$$

$$+ \frac{\mu_4}{h_4} [V_4 - V_0]] + \frac{2\Delta t^2}{h_1\rho_1} \frac{\mu_1}{h_1} (V_1 - V_0) \qquad (3.5.8)$$

with $(\lambda, \mu, \rho)_2 = \frac{1}{2} (\lambda, \mu, \rho)_2^+$

$(\lambda, \mu, \rho)_4 = \frac{1}{2} (\lambda, \mu, \rho)_4^+$

and $(\lambda, \mu, \rho)_1 = (\lambda, \mu, \rho)_2^+ = (\lambda, \mu, \rho)_4^+$

if the media is homogeneous on Γ^+.

3.5.a.3 Quarter plane

The parameter values relevant to the closed domain $\bar{\mathbb{D}}^{II} \cup \bar{\mathbb{D}}^{III} \cup \bar{\mathbb{D}}^{IV}$ are null

i.e.
$$(\lambda, \mu, \rho)_{\bar{1}} = 0$$
$$(\lambda, \mu, \rho)_{2} = 0$$
$$(\lambda, \mu, \rho)_{3} = 0$$
$$(\lambda, \mu, \rho)_{\bar{4}} = 0$$

Equations (3.4.1) to (3.4.3) become

$$\begin{aligned} U^{1} = 2U-U^{-1} &+ \frac{2\Delta t^{2}}{\rho_1 h_1} (\lambda_1+2\mu_1)/h_1 \; [U_1-U_0] \\ &+ \frac{\Delta t^{2}}{2h_1\rho_1} \left\{ \frac{\lambda_1^{+}}{h_2} [(W_6-W_1) + (W_2-W_0)] \right. \\ &+ \frac{2\Delta t^{2}}{h_2\rho_2} \left\{ \frac{\mu_2}{h_2} [U_2-U_0] \right\} \\ &+ \frac{\Delta t^{2}}{2h_2\rho_2} \left\{ \frac{\mu_2^{+}}{h_1} [(W_6-W_2) + (W_1-W_0)] \right\} + O(h^{2}) \end{aligned}$$

...(3.5.9)

$$W^1 = 2W - W^{-1} + \frac{2\Delta t^2}{h_2\rho_2}\frac{(\lambda_2+2\mu_2)}{h_2}(W_2-W_0)$$

$$+ \frac{\Delta t^2}{2h_2\rho_2}\left[\frac{\lambda_2^+}{h_1}\left[(U_0-U_2) + (U_1-U_0)\right]\right.$$

$$+ \frac{2\Delta t^2}{h_1\rho_1}\frac{\mu_1}{h_1}[W_1-W_0]$$

$$+ \frac{\Delta t^2}{2h_1\rho_1}\left\{\frac{\mu_1^+}{h_2}\left[(U_6-U_1) + (U_2-U_0)\right]\right\} + O(h^2)_\Gamma$$

$$V^1 = 2V - V^{-1} + \frac{2\Delta t^2}{h_2\rho_2}\mu_2(V_2-V_0) + \frac{2\Delta t^2}{\rho_1 h_1}\frac{\mu_1}{h_1}(V_1-V_0)$$

$$+ O(h^3)_\Gamma \tag{3.5.10}$$

with $(\lambda, \mu, \rho)_1 = \frac{1}{2}(\lambda, \mu, \rho)_1^+$,

$(\lambda, \mu, \rho)_2 = \frac{1}{2}(\lambda, \mu, \rho)_2^+$.

and $(\lambda, \mu, \rho)_1^+ = (\lambda, \mu, \rho)_2^+$

if the medium is homogeneous on Γ^+.

3.5.a.4 Diagonal stress free boundary

If $(\lambda, \mu, \rho)_3 = 0$

$(\lambda, \mu, \rho)_4 = 0$

the equations (3.4.1) to (3.4.3) become:

$$U^{1} = 2U-U^{-1} + \frac{2\Delta t^{2}}{h_1\rho_1}\frac{\lambda_1+2\mu_1}{h_1}[U_1-U_0]$$

$$+ \frac{\Delta t^{2}}{2h_1\rho_1}\{\frac{\lambda_1^{+}}{h_2}[(W_6-W_1) + (W_2-W_0)]$$

$$+ \frac{\lambda_1^{-}}{h_4}[(W_1-W_5)]$$

$$+ \frac{2\Delta t^{2}}{h_2\rho_2}\frac{\mu_2}{h_2}[U_2-U_0]$$

$$+ \frac{\Delta t^{2}}{2h_2\rho_2}\{\frac{\mu_2^{+}}{h_1}[(W_6-W_2) + (W_1-W_0)]$$

$$+ \frac{\mu_2^{-}}{h_3}(W_2-W_7)] + O(h^{2})_{\Gamma}$$

$$W^{1} = 2W-W^{-1} + \frac{2\Delta t^{2}}{h_2^{2}\rho_2}[\lambda_2+2\mu_2][W_2-W_0]$$

$$+ \frac{\Delta t^{2}}{2h_2\rho_2}\ \frac{\lambda_2^{+}}{h_1}[(U_1-U_0) + (U_6-U_2)]$$

$$+ \frac{\lambda_2^{-}}{h_3}(U_7-U_2)\} + \frac{2\Delta t^{2}}{h_1^{2}\rho_1}\mu_2\ [W_1-W_0]$$

$$+ \frac{\Delta t^{2}}{2h_2h_1\rho_1}\{\mu_1^{-}\ [U_1-U_5] + \mu_1^{+}\ [U_6-U_1]\}$$

$$+ \frac{\Delta t^{2}}{2h_2h_1\rho_3}[\mu_3^{+}\ [U_2-U_0]] + O(h^{3}) \tag{3.5.11}$$

$$V^{1} = 2V-V^{-1} + \frac{2\Delta t^{2}}{h_1^{2}\rho_1}[\mu_1\ [U_1-U_0]$$

$$+ \frac{2\Delta t^{2}}{h_2^{2}\rho_2}\mu_2\ [U_2-U_0] + O(h^{3}) \tag{3.5.12}$$

3.5.b Stress free boundaries when the medium is homogeneous on Γ^+

The expansion consists in a simplification of the previous one. We will consider only the horizontal case since it is the most common and it will be possible to compare the results to our work.

With $$h_1 = h_3\ ,\ h_2 = h_4$$

$$(\lambda,\mu,\rho) = (\lambda,\mu,\rho)_1 = (\lambda,\mu,\rho)_3 = (\lambda,\mu,\rho)_4$$

The schemes (3.5.2), (3.5.3) yield:

$$U^1 = 2U - U^{-1} + \frac{\Delta t^2}{h_1^2\rho}(\lambda + 2\mu)[U_1 - 2U_0 + U_3]$$

$$+ \frac{\Delta t^2}{2h_1h_2}\lambda[(W_6 - W_1) + (W_3 - W_7)]$$

$$+ \frac{2\Delta t^2}{h_2^2\rho}\mu_2[U_2 - U_0]$$

$$+ \frac{\Delta t^2}{2h_1h_2\rho}[\mu_2[(W_6 - W_7) + (W_1 - W_3)]$$

$$0(h^2) \qquad (3.5.13)$$

$$W^1 = 2W - W^{-1} + \frac{2\Delta t^2}{h_2^2 \rho} [\lambda + 2\mu][W_2 - W_0]$$

$$+ \frac{\Delta t^2}{2h_1 h_2 \rho} \lambda_2[(U_6 - U_7) + (U_1 - U_3)]$$

$$+ \frac{\Delta t^2}{2h_1 h_2 \rho} \mu[(U_6 - U_1) + (U_3 - U_7)]$$

$$+ \frac{\Delta t^2}{h_1^2 \rho} \mu[W_1 - 2W_0 + W_3]$$

$$+ O(h^2) \qquad (3.5.14)$$

3.5.c Comparison with "numerical experiment methods"

Ilian et al (1975), Ilian and Lowenthal (1976), Ilian (1978) studied experimentally the stability of different combination of schemes in simple cases.

The finding is that the two classical approximations using fictitious lines are unstable for $\beta/\alpha > .5$.

The range of stability corresponding to different combinations of derivatives lead to the following table (Ilian, 1978).

Centered approximation; Range of stability	$\beta/\alpha > .3$
One sided	$\beta/\alpha > .350$
Composed	$\beta/\alpha > .375$
New composed	$\beta/\alpha > .000$

The last results are normal since the stability is independent of β/α ratio.

We can state that equations (3.5.13), (3.5.14) correspond to the most stable scheme.

3.6 COMPARISON WITH THE CASE WHERE THE BOUNDARIES ARE ASSIMILATED TO THE LIMIT OF THE TRANSITION ZONE ("HETEROGENEOUS MEDIA" FORM)

In the displacement vector case, simple calculus shows that the terms independent of derivatives given by a Taylor development of second derivatives (equations (3.2.3), (3.2.4), (3.2.13), (3.2.14)) are identical to the case of a transition zone when its thickness tends to zero. As in previous chapters if we call these developments F.D(.), equations (3.3.10), (3.3.11) and (3.3.12) can be written:

$$FD(D_{tt}u) = FD(.) + \frac{(\lambda+2\mu)_3 D_x u - (\lambda+2\mu)_1 D_x u}{(h_1\rho_1+h_3\rho_3)}$$

$$+ [\mu_4 D_z u - \mu_2 D_z u]/(h_2\rho_2 + h_3\rho_3) \tag{3.6.1}$$

$$FD(D_{tt}w) = FD(.) + \frac{(\lambda+2\mu)_4 D_z W - (\lambda+2\mu)_2 D_z w}{(h_2\rho_2+h_4\rho_4)}$$

$$+ [\mu_3 D_x u - \mu_1 D_x u]/(h_1\rho_1 + h_3\rho_3) \tag{3.6.2}$$

$$F\ D(D_{tt}v) = F\ D(.) + (\mu_4 D_z v - \mu_2 D_z v)$$

$$+ (\mu_3 D_x v - \mu_1 D_x v) \tag{3.6.3}$$

The boundary conditions (3.3.4) to (3.3.6) yield

$$(\lambda+2\mu)\,{}_3D_x u - (\lambda+2\mu)\,{}_1D_x u = \lambda_1 D_z w - \lambda_3 D_z w \tag{3.6.4}$$

$$(\lambda+2\mu)\,{}_4D_z w - (\lambda+2\mu)\,{}_2D_z w = \lambda_2 D_x u - \lambda_4 D_x u \tag{3.6.5}$$

$$\begin{gathered} \mu_3 D_x w - \mu_1 D_x w = \mu_1 D_z u - \mu_3 D_z u \\ \\ \mu_4 D_z u - \mu_2 D_z u = \mu_2 D_x w - \mu_4 D_x w \end{gathered} \tag{3.6.6}$$

Replacing (3.6.4) to (3.6.6) in equations (3.6.1), (3.6.2) yield

$$\begin{aligned} F\ D(D_{tt}u) = {} & F\ D(.) + 2\Delta t^2 \frac{(\lambda_1 D_z w - \lambda_3 D_z w)}{h_1\rho_1 + h_3\rho_3} \\ & + 2\Delta t^2 \frac{(\lambda_2 D_x w - \lambda_4 D_x w)}{h_2\rho_2 + h_3\rho_3} \end{aligned} \tag{3.6.7}$$

$$\begin{aligned} F\ D(D_{tt}w) = {} & F.D(.) + 2\Delta t^2 \frac{(\lambda_2 D_x u - \lambda_4 D_x u)}{h_2\rho_2 + h_4\rho_4} \\ & + 2\Delta t^2 \frac{(\mu_1 D_z u - \mu_3 D_z u)}{h_1\rho_1 + h_3\rho_3)} \end{aligned} \tag{3.6.8}$$

$$F\ D(D_{tt}v) = F\,D(.) \tag{3.6.9}$$

(since for SH waves $\mu_4 D_z v - \mu_2 D_z v = 0$ and $\mu_3 D_x v - \mu_1 D_x v = 0 \quad \forall v(P) \quad P \in \overline{\Omega}$)

Equations (3.6.7) to (3.6.9) show that

i) if $P(x,z)$ is an inner point the developments are of course identical since the boundary terms cancel.

ii) For SH waves the two methods are identical since the differential terms at the boundary cancel (eqn. 3.6.9). This lets us forecast different results for P.SV waves' potential equations. Although they have the same form, the boundary terms will not vanish.

iii) For the displacement vector equations the error is considerable and equal to

$$\Delta t^2(\lambda_1 - \lambda_3)D_z w/(h_1\rho_1 + h_3\rho_3)$$

$$+ \Delta t^2(\mu_2 - \mu_4)D_x w/(h_2\rho_2 + h_4\rho_4)$$

for the horizontal component u.

and $$\Delta t^2(\lambda_2 - \lambda_4)D_x u/(h_2\rho_2 + h_4\rho_4)$$

$$+ \Delta t^2(\mu_1 - \mu_3)D_z u/(h_1\rho_1 + h_3\rho_3)$$

for the vertical component w.

Therefore we can see that assimilation of the boundary conditions to the limit of a transition zone, although correct in the case of SH waves (Boore 1972), is not acceptable

in the case of a displacement vector P-SV., as in Kelly et al "heterogeneous media" equations (1976) We shall consider later the case of P-SV potential equations since this case has been studied analytically by Gupta (1966).

3.7 INITIAL CONDITIONS AND SOURCE REPRESENTATION

Alterman and Karal (1968) solved the difficulties encountered in the neighborhood of the source by subtracting the displacement due to the source from the total displacement field in a rectangular region surrounding the source. The direct source contributions being given from the known analytical solution for the source in an infinite region, and displacement continuity conditions being applied at the boundary of the rectangular region. This method is also applied by Kelly et al (1976).

Alterman and Aboudi (1970), Aboudi (1971) give a numerical treatment of seismic sources in elastic media which are equivalent to body forces.

We remark that although the problems can be superposed, we shall distinguish two different problems.

3.7.a)

The initial displacement and velocity field are prescribed in the considered region Ω

$$U_0(P) = g(x,0)$$

$$\dot{U}_0(P) = f(x,0)$$

$$\forall P \in \Omega$$

$$\dot{U},\ U \in L^2(\Omega) \qquad (3.7.1)$$

where U_0 is solution of the homogeneous equation of motion

$$AU + \frac{\partial^2 U}{\partial t^2} = 0$$

In this case, no special treatment is necessary (see Boore 1970).

3.7.b)

The displacement is due to a source $S(x,t)$ defined in a region $\overline{\Omega}_s \subset \Omega$ of duration $[0,\tau]$ such that

$$S(P) = 0 \qquad \forall P(x,t) \not\subset \overline{\Omega}_s \times [0,\tau) \quad (3.7.2)$$

which is the common case of seismic exploration.

If $S_p(x,t)$ represents the displacement potential the equations of motion yield

$$Au + \rho D_{tt} u + \rho D_{tt}[\text{grad } S_p(x,t)] = 0 \qquad \overline{\Omega}_s \times [0,\tau) \quad (3.7.3)$$

and the source will act as a body force since it is a body force.

For $S_d(x,t)$ representing the displacement source we shall have

$$AU + \rho D_{tt} U - \rho D_{tt} S_d(x,t) = 0$$

$$\text{in } \Omega \times [0,\tau) \qquad (3.7.4)$$

We note that equations (3.7.3) and (3.7.4) express very clearly the reciprocity between source and receiver.

In the discrete domain (3.7.4) becomes

$$\tilde{A}U + \rho \tilde{D}_{tt} U - \rho \tilde{D}_{tt} S_d(x,t) = 0 \qquad (3.7.5)$$

The development of $D_{tt} S_d(x,t)$ being

$$\rho \tilde{D}_{tt} S_d(x,t) = 2\rho \frac{S_d^1 - S_d^0}{\Delta t^2}$$

$$(\text{in } \overline{\Omega}_s \times [0,\tau)) \qquad (3.7.6)$$

for simple reason of causality.

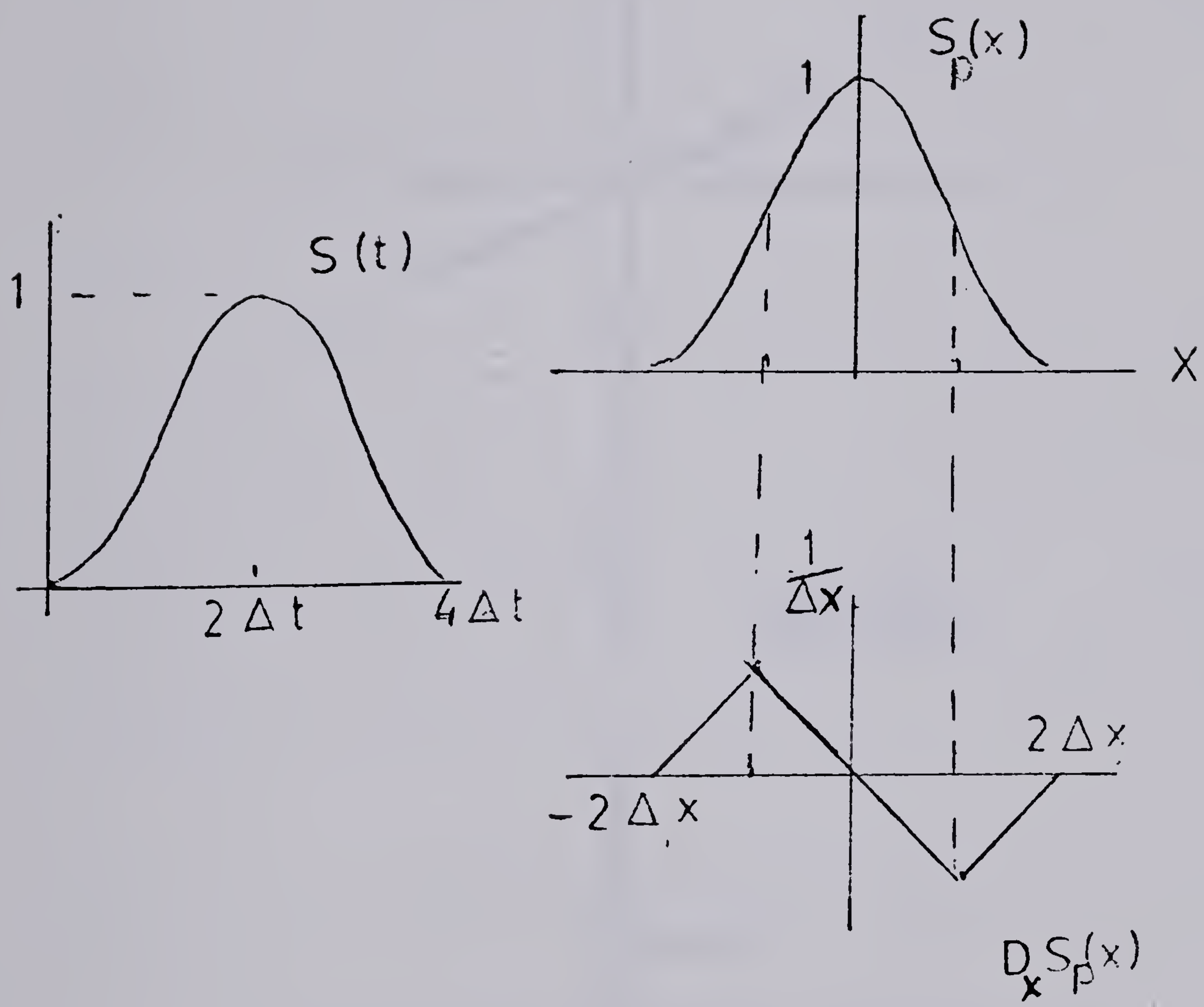
In this case, no restriction has to be imposed on the source function, except that for a numerical treatment S_d has to be band limited, that is its amplitude spectrum has to be non zero over a finite range of the transform variable, to avoid aliasing errors (see next chapter).

Simulation of forces such as defined by Aboudi (1971), Archambeau (1968), Burridge and Knopoff (1964), Burridge, Lapwood and Knopoff (1964) can be applied without complications.

We note that for second order source development (3.7.6) will be

$$\rho\tilde{D}_{tt}S_d(x,t) \sim 2\rho\left[\frac{S_d^1 - S_d^0}{\Delta t^2} - \frac{\dot{S}_d}{\Delta t}\right]$$

where $\dot{S}_d(P),\ S_d(P) = 0$ for $P \not\in \overline{\Omega}_s \times [0,\tau]$



$$K = \ddot{S}(t)\, D_x S_p(x)$$

fig 3.2

Fig. 3.2. Representation of a source with a Gaussian distribution.

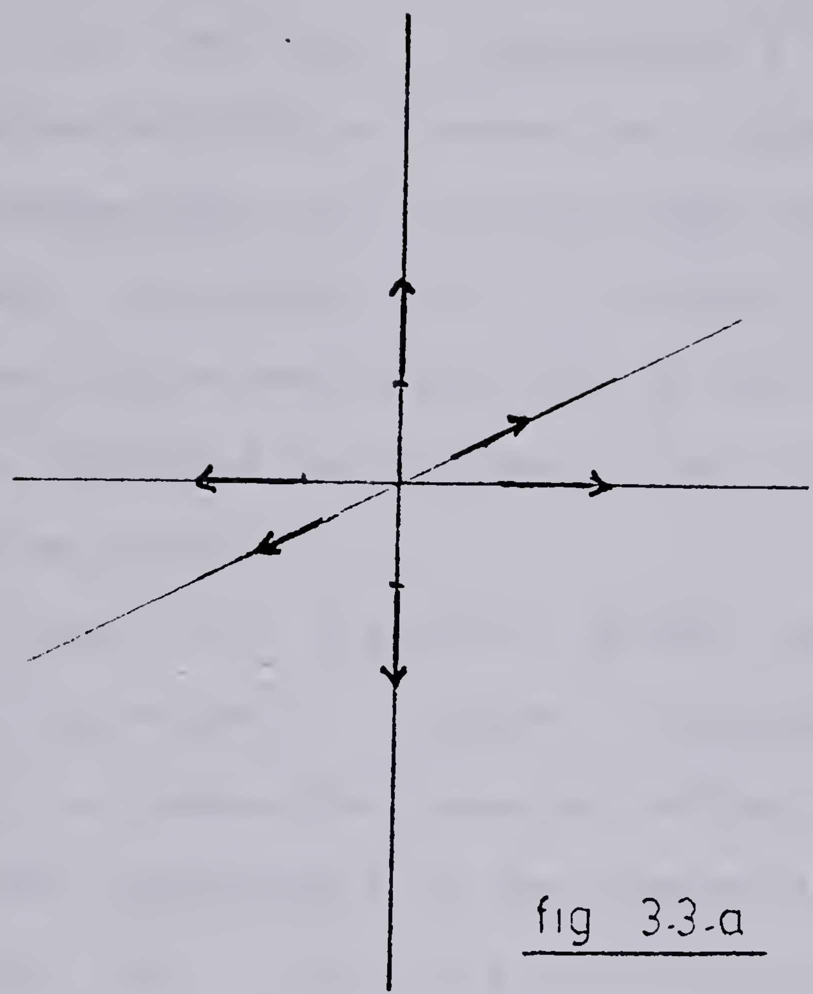


fig 3.3-a

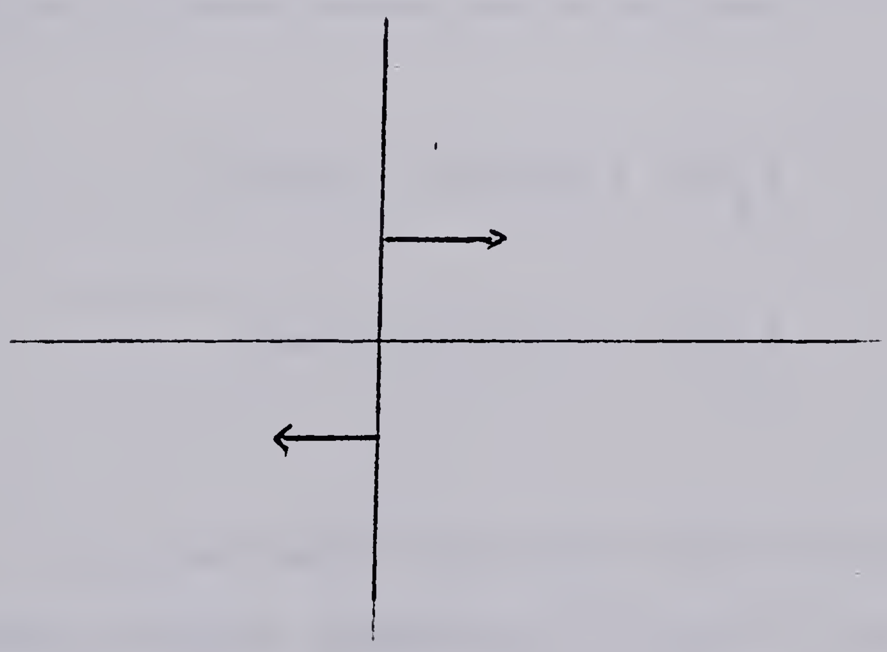


fig 3.3-b

Fig. 3.3. Forces representing a source

3.3.a Dilatation source (superposition of dipoles);

3.3.b Shear source (superposition of couples).

3.8 GRID EDGE BOUNDARIES

To limit the area of computation it is essential to introduce artificial boundaries. Those constraints are of Neumann type and the grid sides behave as perfect reflectors. The problem can, of course, be overcome expensively by widening the grid in such a way that the undesired reflections do not disturb the required solutions.

If there is at present no perfect way to simulate an absorbing boundary, a number of methods have been developed to cancel the unwanted reflections by introducing some constraints at the boundaries. Lysemer and Kuhlemeyer (1969) achieved attenuation at the boundaries by applying viscous tractions such that

$$\text{normal stresses} = a\rho v_p \dot{w}$$

$$\text{shear stresses} = b\rho v_s \dot{u} \qquad (3.8.1)$$

where $\dot{u}$, $\dot{w}$ are as usual the time derivatives of the displacement vector components, v_p and v_s the compressional and shear velocities, and a,b are dimensionless coefficients. Cartellani (1974) used a similar approach where the viscous forces are represented by

$$\sigma_n = -V_p \rho \dot{\vec{U}} \cdot \vec{n}$$

$$\tau_n = -V_s \rho \vec{U} \times \vec{n} \qquad (3.8.2)$$

where σ_n and τ_n are respectively normal and tangent stresses; $\dot{\bar{u}}$ is the velocity vector, $\vec{n}$ and $\vec{t}$ are the unit normal and tangent vector.

Those viscous dash-spots are at the base of most techniques used for absorbing boundaries. The quality of the attenuation is a function of the frequency and angle of incidence of the incident wave (Lysmer and Kuhlemeyer (1969), Castellani (1974), Robinson (1977)).

W.D. Smith (1975) took advantage of the phase difference between free (Neuman) and clamped (Dirichlet) boundaries to attenuate the reflections by adding them. A closer look shows that the method is insufficient for elastic waves since Rayleigh waves do not exist for the clamped boundaries.

The proposed approach is to simulate increasing internal friction in a buffer zone (Voigt Solid, Ewing et al, 1957). Then, the stresses can have the form

$$\sigma^1_{xx} = \sigma_{xx} + a \frac{\partial}{\partial t} \sigma_{xx}$$

$$\sigma^1_{xz} = \sigma_{xz} + b \frac{\partial}{\partial t} \sigma_{xz}$$

If we simplify by writing a = b, we obtain

$$\sigma^1_{xx} = (1 + a\,\frac{\partial}{\partial t})\,\sigma_{xx}$$

$$\sigma^1_{xz} = (1 + a\,\frac{\partial}{\partial t})\,\sigma_{xz}\ . \qquad (3.8.3)$$

Then, the equations of motion become

$$u_{tt} = (1 + a\,\frac{\partial}{\partial t})\,[D_x\sigma_{xx} + D_z\sigma_{xz}] \qquad (3.8.4)$$

$$w_{tt} = (1 + a\,\frac{\partial}{\partial t})\,[\partial_z\sigma_{zz} + \partial_x\sigma_{zx}] \qquad (3.8.5)$$

when the equations (3.8.4), (3.8.5) are a simplified form of Voigt's relation. For further simplification we admit that:

$$a(U^1 - U^{-1})/2\Delta t = -\varepsilon^2 U.$$

$$\tilde{u}_{tt} = (1 - \varepsilon^2)\,[\tilde{D}_x\tilde{\sigma}_{xx} + \tilde{D}_z\tilde{\sigma}_{xz}] \qquad (3.8.6)$$

$$\tilde{w}_{tt} = (1 - \varepsilon^2)\,[\tilde{D}_z\tilde{\sigma}_{zz} + \tilde{D}_x\tilde{\sigma}_{zx}] \qquad (3.8.7)$$

To realize this model it suffices to multiply the equation by a coefficient equal at $k^2 = (1 - \varepsilon^2) < 1$ this coefficient decreasing progressively to avoid reflections in the buffer zone. The effect is the same as increasing

the grid size by $\frac{1}{(1 - \varepsilon^2)^{\frac{1}{2}}} > 1$ or decrease progressively the velocities. Generally 8 to 12 spatial steps are sufficient as a buffer zone.

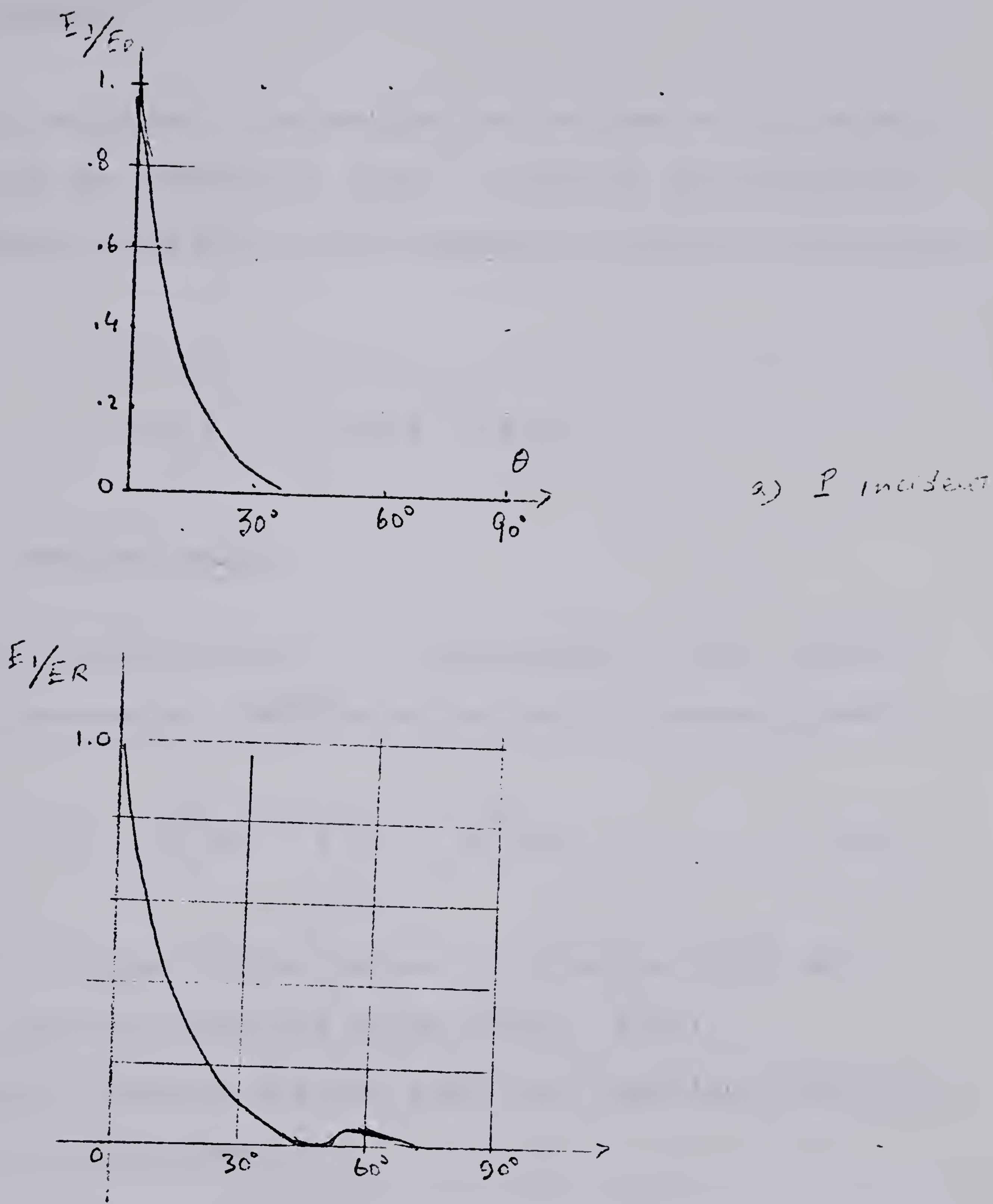


Fig. 3.4. Absorbing boundary conditions (Castellani method)

3.4.a Ratio of reflected and incident energy for compressional waves as functions of angle of incidence

3.4.b Energy ratio for shear waves

(after Castellani 1974)

3.9 ACCURACY

The accuracy is determined by the sum of the rounding error and the truncation error. Let U(P) be the analytical solution and $\tilde{U}(P)$ be the numerical solution at a point P.

$$P(x,z,t) \subset \Omega \subset R^2 \times [0,\tau]$$

3.9.a Rounding error

The rounding error is a consequence of the computational procedure. We have at the initial moment (Smith, 1978)

$$N^0(P) = U^0(P) - r^0(P) \qquad (3.9.1)$$

where r^0 is the initial vector of rounding error and U^0 the initial numerical value ($U^0(P) \subset \tilde{U}(P)$).

If we consider the two time step numerical recursive solution of the form

$$U^j = \tilde{A}U^{j-1} \qquad (3.9.2)$$

then

$$N^1 = \tilde{A}N^0 - r^1$$

$$N^j = \tilde{A}^j U^0 - \tilde{A}^j r^0 - \tilde{A}^{j-1} r^1 - \ldots - r^j \qquad (3.9.3)$$

and

$$U^j - N^j = \tilde{A}^j r^0 + \tilde{A}^{j-1} r^1 + \ldots + r^j \qquad (3.9.4)$$

This shows that the rounding error vector r propagates the same way as the numerical solution vector U.

If Λ_i are the eigenvalues of $\tilde{A}^i$, $i = 1(1)(N - 1)$ then equation (3.9.4) converges if

$$\max|\Lambda_i| < 1.$$

3.9.b Truncation error

Let $$F\ D(\tilde{U}(P)) = 0 \qquad (3.9.5)$$

represent the finite difference equation at $P(x,z,t)$. If U represents the exact solution of the differential equation then the truncation error is defined as

$$\varepsilon(P) = F.D(U(P))$$

$$\text{with } P \subset \bar{\Omega} \subset \mathbb{R}^n \times [0,T] \qquad (3.9.6)$$

Consequently ε is derived from the Taylor development residual terms.

3.9.b.1 Case of P.SV waves. From (3.9.6), (3.3.10) and (3.3.11) we have that

$$\varepsilon = \left|\frac{\Delta}{h^2\rho}(\lambda,\mu)O(h^2)\right|_{\Gamma} + \frac{1}{h^2\rho}(\lambda,\mu)O(h^4) + O(\Delta t^2) \qquad (3.9.7)$$

where $\frac{\Delta}{h^2\rho}(\lambda,\mu)O(h^2)\Big|_{\Gamma} = 0$ if P is an inner point.

Then $\qquad \varepsilon = O(h^2) + O(\Delta t^2) \qquad$ if $P \subset \Omega$

3.9.b.2 Case of SH waves

From equation (3.3.12) we obtain

$$\varepsilon = \frac{\Delta\mu}{h^2\rho}O(h^3)\Big|_{\Gamma} + \frac{\mu}{h^2\rho}O(h^4) + O(\Delta t^2) \qquad (3.9.8)$$

but

$$\frac{\Delta\mu}{h^2\rho}O(h^3)_{\Gamma} = \frac{1}{3!}(\mu_1 - \mu_3)\frac{h^3}{h^2}D_{xxx}V + \frac{1}{3!}(\mu_2 - \mu_4)\frac{h^3}{h^2}D_{zzz}V \qquad (3.9.9)$$

since the boundary conditions can be written

$$\mu_1 D_x^+ U = \mu_3 D_x^- U$$

$$\mu_2 D_z^+ U = \mu_4 D_z^- U \qquad (3.9.10)$$

we have

$$\frac{\Delta\mu}{h^2\rho} \, O(h^3) = 0.$$

Then

$$\varepsilon = O(h^2, \Delta t^2)$$

since (3.9.10) can be written

$$\mu_1 D_x^{(2n+1)+} = \mu_3 D_x^{(2n+1)-}$$

$$\mu_2 D_z^{(2n+1)+} = \mu_4 D_z^{(2n+1)-} \tag{3.9.11}$$

This result shows that for the SH wave equation, and consequently for liquid media, the error truncation order is independent of the boundaries (in the case of stress and displacement continuity). Then, more generally,

$$\varepsilon = \sum_{n=1}^{N} O(h^{2n}, \Delta t^{2n}) \tag{3.9.12}$$

From a Taylor series (3.9.12) can be written

$$\varepsilon = \sum_{n=1}^{N} [D_t^{2n} v \Delta t^{2(n-1)} - \frac{\mu}{\rho} [D_x^{2n} v + D_z^{2n} v] h^{2(n-1)}]$$

$$+ O(\Delta t^{2n}, h^{2n}) \tag{3.9.13}$$

But from the equation of motion

$$D_t^{2n}V = [D_x^2V + D_z^2V]^{(n)}(\frac{\mu}{\rho})^n$$

then

$$\varepsilon = \sum_{n=1}^{N} \frac{\mu}{\rho} \quad (\frac{\mu}{\rho})^{n-1}[D_x^2V + D_z^2V]^{(n)}\Delta t^{2(n-1)}$$

$$- [D_x^{2n}V + D_z^{2n}V]h^{2(n-1)}] \qquad (3.9.14)$$

which lead for an undirectional wave

$$\varepsilon = \sum_{n=1}^{N} \frac{\mu}{\rho} h^{2n-1}[(\frac{\mu}{\rho})^{n-1}(\frac{\Delta t}{h})^{2(n-1)} - 1]D_z^{2n}V \qquad (3.9.15)$$

Then, in the case of an unidirectional wave $\varepsilon = 0$ for any order of development, if

$$\frac{\mu}{\rho}(\frac{\Delta t}{h})^2 = 1 \qquad (3.9.16)$$

This relation is important since it forecasts the identity between the analytical solution and the numerical solution for synthetic seismograms. This fact can be easily verified.

Then in the case of

$$\frac{\mu}{\rho}\left(\frac{\Delta t}{h}\right)^2 = 1$$

we have not only consistency, we have an identity between unidirectional SH waves and their analytical solution.

3.10 CONSISTENCY, STABILITY AND CONVERGENCE

When approximating the analytical problem to a discrete problem one main requirement is when h_i, $\Delta t \to 0$; $\tilde{A}U \to AU$ i.e. the discrete problem becomes equivalent to the continuous one, and the numerical problem is said to be consistent to the analytic problem.

However, the consistency does not necessarily imply that the numerical solution approximate the analytical solution and converges to it.

Although theoretically, consistency and convergence are sufficient conditions, Van Howen (1968); rounding errors give rise to a computational solution instead of the time difference solution. This leads to the stability of a difference scheme.

3.10.a Consistency

The consistency is measured by the truncation error.

Since $\quad F.D(\tilde{U}) = 0 \quad$ (by definition (3.9.6))

$$\varepsilon = \left| F.D(U) - FD(\tilde{U}) \right| \tag{3.10.1}$$

The equations (3.9.7), (3.9.8) shows that if h,

$\Delta t \to 0 \qquad \varepsilon \to 0$

then $\quad F\,D(U) \rightarrow FD(\tilde{U})$

i.e. $\quad F\,D(U) \rightarrow 0$

and $\quad F\,D(U) \rightarrow (A + D_{tt})U = 0 \qquad (3.10.2)$

Hence, the finite difference development of the equations of motion in elastic media trend to the differential equation when h, $\Delta t \rightarrow 0$.

3.10.b Stability

There are two methods of investigating the stability criteria. One uses a finite Fourier series (method of Von Neumann);the other expresses the equations in matrix form and examines the eigenvalues of an associated matrix. We shall use the last method since it takes into account the boundary conditions but first we shall examine the simplest problem.

3.10.b.1 Stability of two time level equations

We have

$$U^{\ell+1} = AU^{\ell} \qquad (3.10.3)$$

U^0 being the initial condition

$$\begin{aligned} U^1 &= AU^0 \\ &\vdots \\ U^{\ell+1} &= A^{\ell+1}U^0 \end{aligned} \tag{3.10.4}$$

The system is stable if

$$||A|| \leq 1$$

3.10.b.2

The system can be written on the form:

$$U^{\ell+1} = \tilde{A}U^{\ell} + \tilde{B}U^{\ell-1} \tag{3.10.5}$$

(since $\tilde{A} = F\ D(A)$).

By analogy with (3.10.3) we consider a vector V^{ℓ} such that

$$V^{\ell} = \begin{vmatrix} U^{\ell} \\ U^{\ell-1} \end{vmatrix} \ . \tag{3.10.6}$$

Then (3.10.3) can be written

$$U^{\ell+1} = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} V^{\ell} \tag{3.10.7}$$

where I is the unitary matrix and the amplification matrix is

$$(C) = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}. \tag{3.10.8}$$

By analogy with (3.10.4), (3.10.7) yield

$$U^{\ell+1} = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^{\ell+1} V^{0} \tag{3.10.9}$$

where $V^0 = U^0$ (initial conditions).

The boundedness of (3.10.9) implies (see Boore 1970)

$$||C|| \leq 1$$

i.e.

$$|\Lambda| \leq 1 \tag{3.10.10}$$

where Λ represents the eigenvalues of (C), i.e.

Λ is solution of

$$\det|(C) - \lambda I| = 0 \tag{3.10.11}$$

From Smith (1978) we introduce the matrix

$$(D) = \begin{pmatrix} I & 0 \\ \frac{I}{\lambda} & I \end{pmatrix} \tag{3.10.12}$$

Since det $|(D)| \neq 0$, we write

$$\det|((C) - \lambda I) \cdot (D)| = 0 \tag{3.10.13}$$

which lead to

$$\det|\Lambda I - (A) - (B)/\Lambda| = 0 \tag{3.10.14}$$

From the explicit form of the equation of motion we have

$$(B) = -I$$

then (3.10.14) yield

$$\det|(A) - \Lambda^1| = 0 \tag{3.10.15}$$

with $\Lambda^1 = \Lambda + 1/\Lambda$

then if Λ is an eigenvalue of (C) so is $1/\Lambda$. Consequently

$$|\Lambda| = 1 \tag{3.10.16}$$

and the stability condition (3.10.10) can be written

$$|\Lambda^{1}| \leq 2 \qquad (3.10.17)$$

The eigenvalues of Λ^{1} are given by the Gershgorin and Brauer theorem which states that if the radius ρ_j is as follows:

$$\rho_j = \sum_{k=1}^{n} |a_{jk}|(1 - \delta_j^k) \qquad (3.10.18)$$

then the eigenvalues of (A) lie in at least one of the circles

$$|z - a_{jj}| \leq \rho_j \qquad (3.10.19)$$

in the complex z plane. The operator $\grave{A}U$ is equal to $(\tilde{A} + 2I)U$ where $\tilde{A}$ is the F.D. of the elasticity operator. Then

$$\grave{A}U = \begin{pmatrix} \Delta t^2(\alpha^2\tilde{D}_{xx} + \beta^2\tilde{D}_{zz}) + 2 & (\lambda + \mu)\Delta t^2\tilde{D}_{xz} \\ (\lambda + \mu)\Delta t^2\tilde{D}_{xz} & 2 + \Delta t^2(\alpha^2\tilde{D}_{zz} + \beta^2\tilde{D}_{xx}) \end{pmatrix} \vec{U}$$

$$\ldots(3.10.20)$$

where

$$\alpha^2 = (\lambda + \mu)/\ell$$

$$\beta^2 = \mu/\ell$$

With the Gershgorin theorem we obtain

$$\left|2 - 4 \frac{\Delta t^2}{h^2} (\alpha^2 + \beta^2)\right| \leq |\Lambda^1| \leq 2 \ . \qquad (3.10.21)$$

Then the stability condition is given by

$$\frac{\Delta t^2}{h^2} \leq \frac{1}{\max(\alpha^2 + \beta^2)} \qquad (3.10.22)$$

or

$$\frac{\alpha \Delta t}{h} \leq \frac{1}{(1 + \frac{\beta^2}{\alpha^2})^{\frac{1}{2}}} \qquad (3.10.23)$$

We note that in the case of SH wave the stability condition becomes

$$\frac{\Delta t}{h} \leq \frac{1}{\beta \sqrt{n}} \qquad (3.10.24)$$

where n is the number of spatial dimensions n = 1, 2, 3

3.10.c Convergence

There is convergence if the solution of the finite development tends to the solution of the differential equation when $\Delta t,\ h \to 0$.

i.e.

$$\delta_e = |U - \tilde{U}| \to 0 \text{ when } \Delta t,\ h \to 0$$

For the wave equation case the discretization error is given by

$$\delta_e = \Delta t^2 \varepsilon$$

where ε is the truncation error.

We note that the finite development of the elasticity equation is convergent since stability and consistency implies convergence (Lax Equivalence Theorem; Richtmeyer and Morton (1967)).

3.11 ALIASING AND GRID DISPERSION

3.11.a Discretization of the spatial waveform implies an "effective" sampling at the node. The condition is therefore that the frequency domain where the amplitude spectrum of a function F is non zero is finite (i.e. F has to be band limited).

The relationship between the temporal or spatial grid and the frequency or wave number is given by the sampling theorem (Bracewell, 1965; Kanasewich, 1975) which states that

$$h_i \leq \frac{1}{2K_c}$$

where $$\mathcal{F}\{n_i \to K_i\} = 0 \text{ if } |K_i| > K_c \tag{3.11.1}$$

It is therefore recommended when we have a source with a broad spectrum, to reduce the effective frequency domain by convolving by a function of the form $\frac{\sin x}{x}$ or $(\frac{\sin x}{x})^2$, the amplitude spectra of these functions being the well known box car or triangle (Bracewell, 1965).

Since, in seismic prospecting the incidence angle is close to the vertical, the apparent wave number along the x axis is equal to

$$K_1 = K \sin \theta$$

which implies a wide grid spacing in the x direction if the economic consideration (computer time) surpasses the interest in surface waves.

Generally, as a rule of thumb, eight to ten samples per wave length are considered minimum to approximate the wave field (Boore, 1972; Claerbout, 1976).

3.11.b Dispersion

The dispersion appears as a variation of phase velocity and group velocity as function of the frequency. Consequently, while the low frequency part of the signal will travel with the velocity $v_0 = V_g = V$, the undersampling will affect the high frequency part of a signal whose group and phase velocity will be $V_g < V < V_0$. Then an apparent attenuation will be introduced as well as a delay in phase resulting in a tailing effect.

To investigate these effects we will follow Brillouin (1953) and Alford et al (1974) in the case for SH waves.

3.11.b.1 SH wave dispersion

Let us consider a harmonic plane wave of the form

$$U = U_0 \ell^{i\,(\omega t - k_1 x - k_2 z)} \qquad (3.11.2)$$

where $$K_2 = K \cos \theta$$

$$K_1 = K \sin \theta \qquad (3.11.3)$$

The propagation is governed by

$$\frac{\ddot{U}}{V_0^2} = \nabla^2 U \qquad (3.11.4)$$

Second order finite difference development of the above equations lead to

$$\frac{U^{\ell-1} - 2U + U^{\ell+1}}{V_0 \Delta t^2} = \frac{U_{m-1} - 2U + U_{m+1}}{h^2}$$

$$+ \frac{U_{n-1} - 2U + U_{n+1}}{h^2} \qquad (3.11.5)$$

where $U(x,z,t) = U(mn,\ nh,\ \ell\Delta t)$

To simplify the notation $U(m,n,\ell)$ is represented by U and $U_{(m-1,n,\ell)} = U_{m-1}$, etc.

If $P = \frac{V_0 \Delta t}{h}$ we obtain, with $P \leq \sqrt{2}/2$ (stability condition), after substituting (3.11.2) into (3.11.5):

$$\sin^2 \frac{\omega \Delta t}{2} = P^2 [\sin^2 \frac{K_1 h}{2} + \sin^2 \frac{K_2 h}{2}] \qquad (3.11.6)$$

If G is the number of samples per wave length V and V_g the phase and group velocities, we can write

$$Kh/2 = \pi/G .$$

Then

$$V/V_0 = \frac{\omega}{KV_0} = \frac{G}{P} \sin^{-1}\{P[\sin^{-2} \frac{(\pi \cos \theta)}{G} + \sin^2 \frac{(\pi \sin \theta)}{G}]\}^{\frac{1}{2}}$$

...(3.11.7)

and the group velocity will be obtained by differentiating (3.11.6) with respect to K.

$$\frac{V_G}{V_0} = [\sin(\frac{\pi}{G} \cos \theta)\cos(\frac{\pi}{G} \cos \theta)\cos \theta$$

$$+ \sin(\frac{\pi}{G} \sin \theta)\cos(\frac{\pi}{G} \sin \theta)\sin \theta]/$$

$$\{[1 - P^2\sin^2(\frac{\pi}{G} \cos \theta) - P^2\sin^2(\frac{\pi}{G} \sin^2 \theta)]^{\frac{1}{2}}.$$

$$[\sin^2(\frac{\pi}{G} \cos \theta) + \sin^2(\frac{\pi}{G} \sin \theta)]^{\frac{1}{2}}\} \qquad (3.11.8)$$

If the propagation is parallel to the grid $\theta = 0$ and (3.11.7), (3.11.8) yield:

$$V_P/V_0 = \frac{G}{P} \sin^{-1}[P \sin \frac{\pi}{G}] \tag{3.11.9}$$

$$V_G/V_0 = \frac{\cos \frac{\pi}{G}}{(1 - P^2 \sin^{-2} \frac{\pi}{G})^{\frac{1}{2}}} \tag{3.11.10}$$

3.7.b.2 P.SV Dispersion

In the absence of body forces (2.1.1) yield:

$$\tilde{A}\vec{U}+\rho\tilde{D}_{tt}\vec{U} = 0 \tag{3.11.11}$$

with

$$\tilde{A} = \begin{pmatrix} (\lambda+2\mu)\tilde{D}_{xx}+\mu\tilde{D}_{zz} & (\lambda+\mu)\tilde{D}_{xz} \\ (\lambda+\mu)\tilde{D}_{xz} & (\lambda+2\mu)\tilde{D}_{zz}+\mu\tilde{D}_{xx} \end{pmatrix} \tag{3.11.12}$$

$\tilde{D}_{xx}$, $\tilde{D}_{zz}$, $\tilde{D}_{xz}$ being the finite difference operator of the corresponding derivatives.

As in section 2.7.b.1 we can write for a harmonic plane wave with $\Delta x = \Delta z = h =$ constant

$$\tilde{D}_{xx}U = \frac{1}{h^2}[U_1 - 2U + U_3]$$

$$\tilde{D}_{xx} = 2[1 - \cos K_1 h]/h^2$$

$$= \frac{4}{h^2}\sin^2 K_1 \frac{h}{2} \tag{3.11.13}$$

$$\tilde{D}_{zz} = \frac{4}{h^2}\sin^2 K_2 \frac{h}{2} \tag{3.11.14}$$

$$\tilde{D}_{tt} = 4\sin^2 \frac{\omega\Delta t}{2}/\Delta t^2 \tag{3.11.15}$$

$$\tilde{D}_{xz} = \frac{1}{2h^2}[\cos[K_1 h + K_2 h] - \cos[K_1 h - K_2 h]]$$

$$\tilde{D}_{xz} = -\frac{\sin}{h^2}(K_1 h)\sin(K_2 h) \tag{3.11.16}$$

for convenience we write (3.11.12)

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} \tag{3.11.17}$$

with

$$\tilde{a}_{21} = \tilde{a}_{12} \tag{3.11.18}$$

(3.11.11) has the form

$$\tilde{A}\vec{U} - \Lambda\vec{U} = 0 \tag{3.11.19}$$

or

$$\begin{pmatrix} \Lambda - a_{11} & a_{12} \\ a_{21} & \Lambda - a_{22} \end{pmatrix} \tag{3.11.19^1}$$

which has a characteristic equation

$$\Lambda^2 - \Lambda(\tilde{a}_{11} + \tilde{a}_{22}) + \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}^2 = 0 \tag{3.11.20}$$

The roots being given by

$$\Lambda_1 = \frac{(\tilde{a}_{11} + \tilde{a}_{22}) + \sqrt{\Delta}}{2} \tag{3.11.21}$$

$$\Lambda_2 = \frac{(\tilde{a}_{11} + \tilde{a}_{22}) - \sqrt{\Delta}}{2} \tag{3.11.22}$$

where the discriminant

$$\Delta = (\lambda + \mu)^2[(\tilde{D}_{xx} - \tilde{D}_{zz})^2 + 4(\tilde{D}_{xz})^2] \tag{3.11.23}$$

if we write

$$S_1 = \sin K_1 h/2$$

$$S_2 = \sin K_2 h/2$$

(3.11.23) yield

$$\Delta = \frac{(\lambda+\mu)^2}{h^2}[(4S_1^2 - 4S_2^2)^2 + 4(S_1S_2)^2[1 + S_1^2S_2^2 - (S_1^2 + S_2^2)]$$

or

$$\Delta = 4^2 \frac{(\lambda+\mu)^2}{h^2}[(S_1^2 - S_2^2)^2 + 4(S_1S_2)^2[1 + S_1^2S_2^2 - (S_1^2 + S_2^2)]$$

then

$$\Delta = 4^2 \frac{(\lambda+\mu)^2}{h^2}[(S_1^2 + S_2^2 \pm 2S_1^2S_2^2)]^2$$

and $$\sqrt{\Delta} = \pm 4 \frac{(\lambda+\mu)}{h^2}[S_1^2 + S_2^2 - 2S_1^2S_2^2]^2 \qquad (3.11.24)$$

then

$$\Lambda_1 = \{2(\lambda + 3\mu)(S_1^2 + S_2^2) + 2(\lambda + \mu)[S_1^2 + S_2^2 - 2S_1^2S_2^2]\}/h^2$$

$$\begin{cases} \Lambda_1 = (4(\lambda + 2\mu)(S_1^2 + S_2^2) - 4(\lambda + \mu)S_1S_2)/h^2 & (3.11.25)\text{a} \\ \\ \Lambda_2 = (4\mu(S_1^2 + S_2^2) + 4(\lambda + \mu)S_1S_2)/h^2 & (3.11.25)\text{b} \end{cases}$$

But $\rho\tilde{D}_{tt}U$, $\rho\tilde{D}_{tt}W$ are eigenvalues and

$$\rho\tilde{D}_{tt}U = \frac{4}{\Delta t^2}\,\rho\,\sin^2(\omega_1\,\Delta t/2)$$

$$\rho\tilde{D}_{tt}W = \frac{4}{\Delta t^2}\quad\sin^2(\omega_2\,\Delta t/2) \qquad (3.11.26)$$

then (3.11.25) and (3.11.26) yield

$$\sin^2(\omega_1\,\Delta t/2) = (\lambda+2\mu)\Delta t^2/\rho h^2[S_1^2 + S_2^2] - [\lambda+\mu]\Delta t^2/\rho h^2 S_1S_2 \qquad \ldots(3.11.27)$$

$$\sin^2(\omega_2\,\Delta t/2) = \mu\Delta t^2/\rho h^2 + (\lambda+\mu)\Delta t^2/\rho h^2\; S_1S_2. \qquad (3.11.28)$$

Recall that

$$K_1h/2 = \frac{\pi}{G}\cos\theta$$

$$K_2h/2 = \frac{\pi}{G}\sin\theta\;, \qquad (3.11.29)$$

As usual, we set

$$\alpha^2 = (\lambda + 2\mu)/\rho$$

$$\beta^2 = \mu/\rho \quad ,$$

Then the ratio between the numerical phase velocity and the media velocities are:

$$\alpha_P/\alpha_0 = \frac{\omega_1}{|K|\alpha_0} \tag{3.11.30}$$

$$\beta_P/\beta_0 = \frac{\omega_2}{|K|\beta_0} \tag{3.11.31}$$

This choice is a consequence of the fact that

$$\Lambda_1 - \Lambda_2 = 4(\lambda + \mu)(S_1 - S_2)^2$$

i.e. $\lambda_1 > \lambda_2$

Taking into account (3.11.29) the expressions (3.11.30), (3.11.31) can be written:

$$\alpha_P/\alpha_0 = \omega_1 hG/2\pi\alpha_0 \tag{3.11.32}$$

$$\beta_P/\beta_0 = \omega_2 hG/2\pi\beta_0 \tag{3.11.33}$$

Since, as mentioned above, $\tilde{D}_{tt}U$ and $\tilde{D}_{tt}W$ are eigenvalue equation (3.11.25) yields, after setting $P = \frac{\alpha_0 \Delta t}{h}$, $\gamma = \beta/\alpha$

$$\sin^2(\omega_1 \ \Delta t/2) = P^2[S_1^2 + S_2^2] - [1 - \gamma^2]P^2 S_1 S_2 \quad (3.11.34)$$

$$\sin^2(\omega_2 \ \Delta t/2) = \gamma^2 P^2[S_1^2 + S_2^2] + [1 - \gamma^2]P^2 S_1 S_2. \quad (3.11.35)$$

Then

$$\omega_1 = \frac{2}{\Delta t} \sin^{-1} P[(S_1^2 + S_2^2) - (1 - \gamma^2)S_1 S_2]^{\frac{1}{2}} \quad (3.11.36)$$

$$\omega_2 = \frac{2}{\Delta t} \sin^{-1} P[(S_1^2 + S_2^2)\gamma^2 + (1 - \gamma^2)S_1 S_2]^{\frac{1}{2}} \quad (3.11.37)$$

The dimensionless phase velocity ratio is then given by

$$\alpha_P/\alpha_0 = \frac{G}{\pi P} \sin^{-1} P[(S_1^2 + S_2^2) - (1 - \gamma^2)S_1 S_2]^{\frac{1}{2}} \quad (3.11.38)$$

$$\beta_P/\beta_0 = \frac{G}{P\pi\gamma} \sin^{-1} P[\gamma^2(S_1^2 + S_2^2) + (1 - \gamma^2)S_1 S_2]^{\frac{1}{2}} \quad \ldots(3.11.39)$$

and the group velocity ratio is given by

$$\alpha_G/\alpha_0 = \frac{1}{\alpha_0} \frac{\partial \omega_1}{\partial K} \quad (3.11.40)$$

$$\beta_G/\beta_0 = \frac{1}{\beta_0}\frac{\partial\omega_2}{\partial K} \tag{3.11.41}$$

We note that for $\gamma = 1$ or $\theta = 0$ the results are the same as for the SH wave or the scalar wave equation given by Alford et al, the dispersion being minimal for values of P chosen at the stability limit.

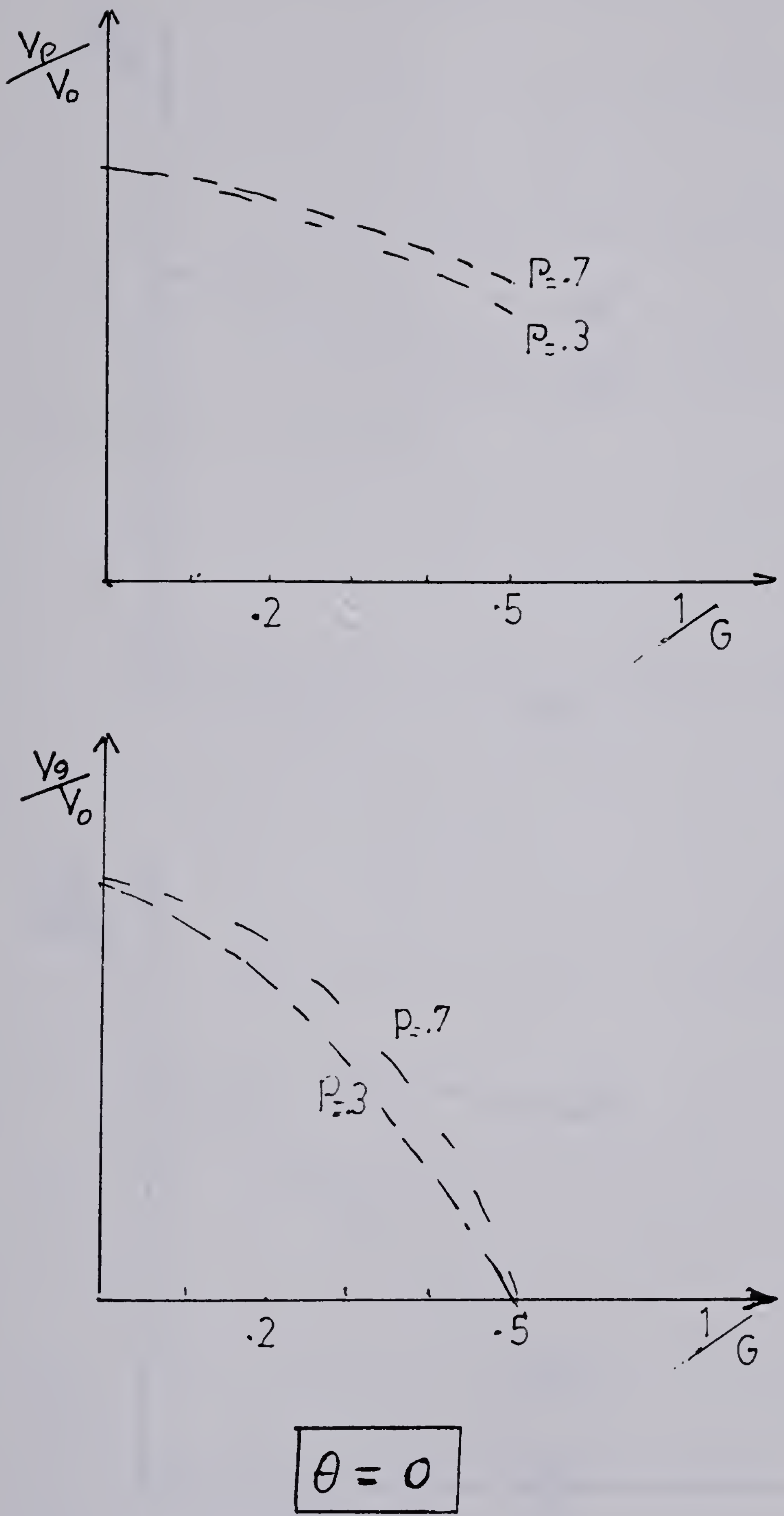
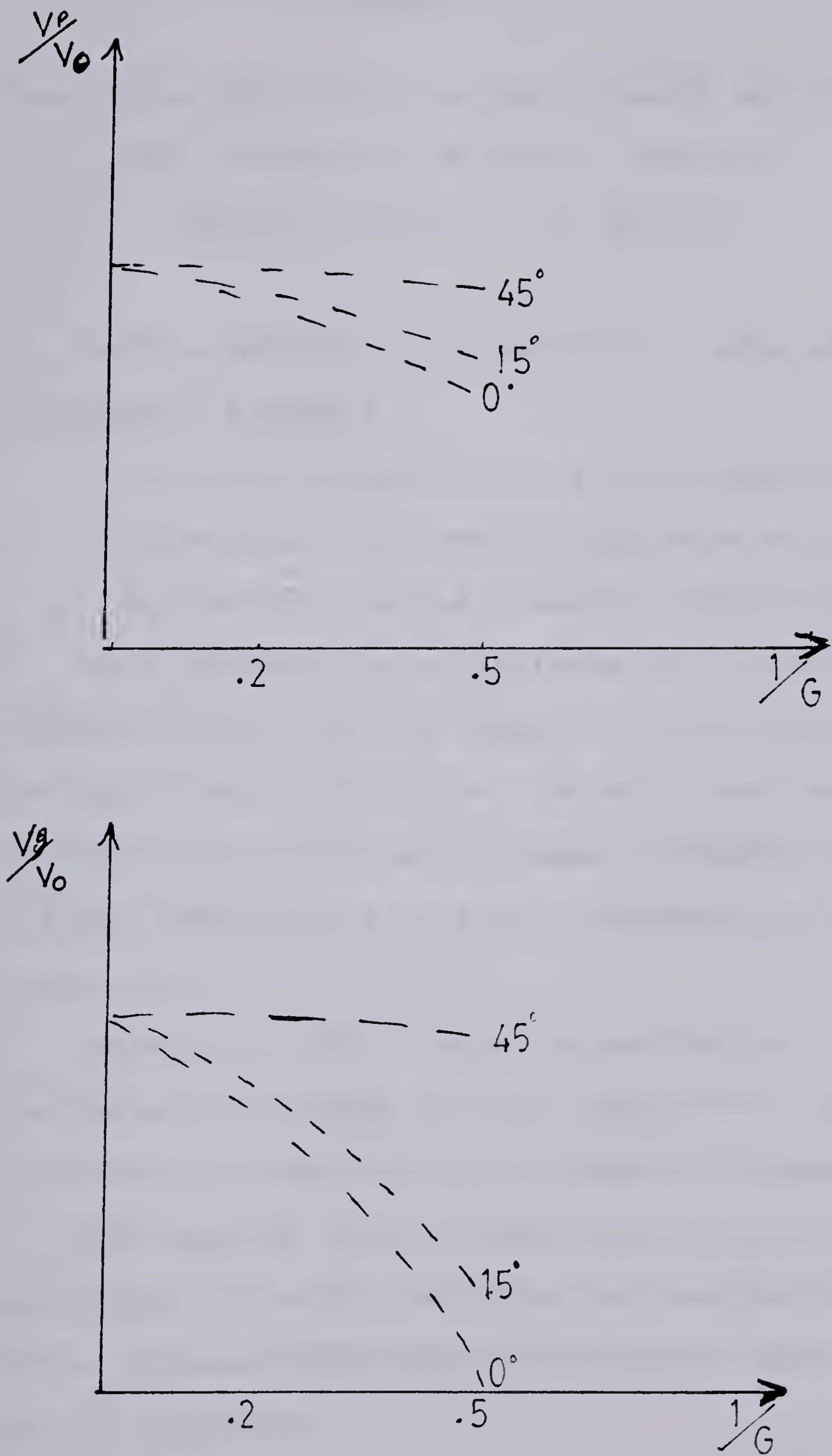


Fig. 3.5 Dispersion curves

3.5.a Normalised phase and group velocity for a propagation parallel to the grid θ = 0



$$P = \frac{V_0 \Delta t}{h} = .7$$

3.5.b Normalised phase and group velocity at different angle of incidence

CHAPTER IV

Comparison with Recursive Developments Derived from Variational Methods - Case of Potential Equations of Motion

We have mentioned in Chapter II, three main methods of numerical analysis:

- The direct expansion by a Taylor series
- The Rayleigh-Ritz method (application of Dirichlet principle).
- The Galerkin method (usually called the variational method

After showing the equivalence of the two last methods we shall develop in this chapter a finite development derived from variational principles. We will then compare it with the finite difference scheme of Chapter II obtained by direct application of Taylor expansion on the usual formulation.

Friedrich (1962), Friedrich and Keller (1966) considered the problem for the operator, $-\nabla^2$, and proceeded by numerical integration in triangular elements.

The approach used in these pages will be more operational, in such a way that we take full advantage of the bilinear forms obtained by use of the z transform and its conjugate.

Use of Sobolev space, W_2^1 will give great advantage for the device of a trial function.

In the second part of this chapter, we will consider the equations of motion (potential) and compare their finite development. The case of the boundary, considered as a limit of a transition zone, will also be considered for those equations.

As in Chapter II, we will consider the equation of motion in the form

$$AU(P) = F(P) \tag{4.1.1}$$

where A is positive definite

$U \in W_2^1$ (Sobolev space)

$P \in \Omega \subset \mathbb{R}^2 \times [0,T)$

According to the Ritz method, we have to minimize the functional

$$I(U) = (AU,U) - 2(U,F)$$

$$= \int_{\Omega} [U(P)AU - 2U(P)F(P)]d\Omega \tag{4.1.2}$$

U(P) represents the acceptable functions from the field of definition D_A of the operator A.

Since for

$$I(U) = \min[(AU,U) - 2(U,F)], \qquad (4.1.3)$$

U is the solution of 4.1.1.

Then, the problem (4.1.2) can be written:

$$(AU,U) - (F,U) = 0 \qquad (4.1.4)$$

which is the Galerkin form.

4.1 FINITE DEVELOPMENT OF THE EQUATIONS OF MOTION DISPLACEMENT BY VARIATIONAL PRINCIPLE

4.1.a Dirichlet Integral Form

After substituting the elasticity operator by its value, (4.1.4) yields:

$$-\int_{\Omega} \{D_x[[\lambda+2\mu]D_xU + \lambda D_zW]U + D_z[\mu D_zU + \mu D_xW]U\}d\Omega$$

$$-\int_{\Omega} F.Ud\Omega = 0 \qquad (4.1.5)$$

$$-\int_{\Omega} \{D_z[[\lambda+2\mu]D_zW + \lambda D_xU]U + D_x[\mu D_xW + \mu D_zU]U\}d\Omega$$

$$-\int_{\Omega} F.Wd\Omega = 0 \qquad (4.1.6)$$

By Gauss Theorem (4.1.5) and (4.1.6) yields

$$\int_{\Omega} \{(\lambda+2\mu)(D_x U)^2 + \lambda D_z W D_x U + \mu(D_z U)^2 + \mu D_x W D_z U\} U d\Omega$$

$$- \int_{\Gamma} [(\lambda+2\mu) D_x U + \lambda D_z W] U . d\Gamma$$

$$- \int_{\Gamma} \mu[D_z U + D_x W] U . d\Gamma - \int_{\Omega} F . U d\Omega = 0 \quad (4.1.7)$$

$$\int_{\Omega} \{(\lambda+2\mu)(D_x W)^2 + \lambda D_x U D_z W + \mu(D_x W)^2 + \mu D_z U D_x W\} W d\Omega$$

$$- \int_{\Gamma} [(\lambda+2\mu) D_z W + \lambda D_x U] W . d\Gamma$$

$$- \int_{\Gamma} \mu[D_x W + D_z U] W . D\Gamma - \int_{\Omega} F . W d\Omega = 0 \quad (4.1.8)$$

If we express the continuity of displacement and stresses at the boundary (Boundary conditions) the integrants of $\int_{\Gamma} (.) d\Gamma$ becomes

$$\int_{\Gamma} (.) d\Gamma = 0 \quad (4.1.9)$$

and (4.1.7) and (4.1.8) yields:

$$\int_\Omega \{(\lambda+2\mu)(D_xU)^2 + \lambda D_zWD_xU + \mu(D_zU)^2 + \mu D_xWD_zU\}Ud\Omega$$

$$- \int_\Omega F.Ud\Omega = 0 \qquad (4.1.10)$$

$$\int_\Omega \{(\lambda+2\mu)(D_zW)^2 + \lambda D_xUD_zW + \mu(D_xW)^2 + \mu D_zWD_xU\}Ud\Omega$$

$$- \int_\Omega F.Ud\Omega = 0 \qquad (4.1.11)$$

4.1.b Discretization and Choice of a Trial Function

Since the objective is the comparison with the finite difference development, it is normal that we choose a trial function from the same field of definition .

Then we have

$$U \in W_2^1 \text{ (Sobolev space)} \qquad (4.1.12)$$

which implies the conservation of norm

$$||U||_{W_2^1(\Omega)} = \left(||U||^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left|\left|\frac{\partial U}{\partial x_i}\right|\right|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \qquad (4.1.13)$$

Then (4.1.12) implies that the function and its first derivative has to have respectively the same expression as the finite difference expression.

As in Chapter III, we consider an elementary region

$$\Omega_h = \mathbb{D}$$

such that

$$\mathbb{D} = \bigcup \mathbb{D}^i .$$

Then, for instance,

$$D_x U = \frac{1}{h_1} [U_0 - U_3]$$

$$\text{for } U \in \mathbb{D}^{II} \cup \mathbb{D}^{III} \text{ etc.} \qquad (4.1.14)$$

If we consider U such that

$$U = U_0 e^{i(\omega t - K_1 m h_1 - K_2 n h_2)}$$

we can use the z transform

with $$z = e^{-iK_2 h_2}$$

$$x = e^{-iK_1 h_1}$$

$$T = e^{i\omega\Delta t}$$

then
$$U_{m,n-1}^{\ell} = UZ$$

$$U_{m-1,n}^{\ell} = UX$$

$$U_{m,n}^{\ell+1} = UT$$

Then (4.1.14) can be written

$$D_x U = \frac{U}{h_1}(1 - Z) \text{ for } U \in \mathbb{D}^{II} \cup \mathbb{D}^{III} \qquad (4.1.15)$$

and

$$D_x U = \frac{UZ^{-1}}{h_1}(1 - Z) \text{ for } U \in D^{I} \cup D^{IV} \text{ etc.} \qquad (4.1.16)$$

For more simplicity we will have $h_1 = h_3$, $h_2 = h_4$, without loosing the generality of our problem.

4.1.c Bilinear form of the expression $(D_x U)^2$, $(D_z U)^2$,... $D_x W D_z U$, $D_z W D_x U$

Using the z transform, we can write

$$(D_x U)^2 = \frac{U_0^2}{h_1^2}[1 - X][1 - X]^*$$

$$U \in \mathbb{D} \qquad (4.1.17)$$

then

$$(D_xU)^2 = \frac{U_0}{h_1^2}\,[\underset{\in D^{II}\cup D^{III}}{(1 - X]U_0} - \underset{\in D^{IV}\cup D^{I}}{X*[1 - X]U_0}] \qquad (4.1.18)$$

and according to section (3.2) if we consider the parameter μ for instance

$$\mu(D_xU)^2 = \frac{U_0}{h_1^2}\,[\mu_3[1 - X]U_0 - \mu_1X*[1 - X]U_0] \qquad (4.1.19)$$

when
$$\mu_1 = (\mu_1^+ + \mu_1^-)/2$$

$$\mu_3 = (\mu_3^+ + \mu_3^-)/2$$

then

$$\mu(D_xU)^2 = -\frac{U}{h_1^2}\,[\mu_1[U_1 - U_0] - \mu_3[U_3 - U_0]] \qquad (4.1.20)$$

By the same way

$$(D_zU)^2 = \frac{U_0}{h_2^2}\,[1 - Z][1 - Z]* \qquad (4.1.21)$$

and

$$\mu(D_zU)^2 = -\frac{U_0}{h_2^2}\,[(1 - Z)\mu_2U_0 - Z*[1 - Z]\mu_4U_0] \qquad (4.1.22)$$

then

$$\mu(D_z U)^2 = -\frac{U_0}{h_2^2}[\mu_2[U_2 - U_0] - \mu_4[U_4 - U_0]] \tag{4.1.23}$$

To calculate the expression $D_x U D_z U$ we have to note that

$$\mathbb{D}(D_x U D_z U)^I = (X,Z) \cap D^I$$

which implies

$$\mathbb{D}(D_x U)^I \subset (X,Z) \cap D^I$$

and

$$\mathbb{D}(D_z U) \subset (X,Z) \cap D^I$$

then

$$D_x U = \frac{U_0}{2}[(1 - X) + Z[1 - X]] \tag{4.1.24}$$

$$D_z U = \frac{U_0}{2}[[1 - Z) + X(1 - Z]] \tag{4.1.25}$$

then

$$D_x W D_z U = \frac{U_0 W_0}{4h_1 h_2}[[1 - X] + 2(1 - X]][(1 - Z) + X(1 - Z)]^*$$

...(4.1.26)

$$D_x W D_z U = \frac{U_0}{4h_1 h_2} [(1 - X] + Z(1 - X]][(1 + X^{-1}] - Z^*[1 + X^{-1}]] \quad \ldots(4.1.27)$$

and referring to section 3.2:

$$\mu D_x W D_z U = \frac{W_0 U_0}{4h_1 h_2} [(1 - X] + Z(1 - X][[1 + X^{-1}]\mu_4 - Z^*[1 + X^{-1})] \quad \ldots(4.1.28)$$

By expliciting (4.1.28) we obtain

$$\begin{aligned} \mu D_x W D_z U = - \frac{U_0}{4h_1 h_2} \{ & [[W_6 - W_3] + [W_1 - W_0]]\mu_2^+ \\ & + [(W_2 - W_7) + (W_0 - W_3)]\mu_2^- \\ & - [(W_1 - W_0) + (W_5 - W_4)]\mu_4^+ \\ & - [(W_0 - W_3) + (W_4 - W_8)]\mu_4^- \} \end{aligned} \quad (4.1.29)$$

In the same way we have

$$D_z W D_x U = \frac{U W_0}{4h_1 h_2} [(1 - Z) + X(1 - Z)][(1 - X] + Z[1 - X]]^* \quad (4.1.30)$$

and

$$\mu D_z W D_x U = \frac{UW_0}{4h_1 h_2} \left[(1 - z) + x[1 - z][[1 + z^{-1}]\mu_3 \right.$$

$$\left. - x^*[1 + z^{-1}]\mu_1 \right] \tag{4.1.31}$$

Explicitly (4.1.31) yields

$$\mu D_z W D_x U = - \frac{U}{4h_1 h_2} \{[(W_6 - W_1) + [W_2 - U_0]]\mu_1^+$$

$$+ [(W_2 - W_0) + (W_7 - W_3)]\mu_3^+$$

$$- [(W_1 - W_5) + (W_0 - W_4)]\mu_1^-$$

$$- [(W_0 - W_4) + (W_3 - W_8)]\mu_3^-\}. \tag{4.1.32}$$

As an abbreviation we will call U.F V(•) the development obtained by finite variational method, in this chapter.

We remark from equation (4.1.28) and (4.1.32) that

$$FV[\mu D_z W D_x U] = F.V(\mu D_x W D_z U]$$

$$\forall P \in \Omega$$

$$\text{but } P \not\in \Gamma$$

Comparing with the results obtained in Chapter III, we have

$$UF\,D\{\xi D_{xx}U\} = -UFV\{\xi D_{xx}U\}$$

$$UF\,D\{\xi D_{zz}U\} = -UFV\{\xi D_{zz}U\}$$

$$\forall P \in \bar{\Omega} = (\Omega + \Gamma) \tag{4.1.33}$$

$$UFD\{\xi D_{xz}W\} = -UFV\{\xi D_x W D_z U\}$$

$$\forall P \in \Omega \tag{4.1.34}$$

and $\quad P \not\in \Gamma.(\vec{n}.0\vec{z})$

$$UFD\{\xi D_{xz}W\} \neq -UFV\{\xi D_x W D_z U\}$$

if $\quad P \in \Gamma.(\vec{n}.0\vec{z}) \qquad (4.1.35)$

when $\vec{n}.0\vec{z}$ design the projection of the normal of the boundary to the 0z axis.

In this case if $\vec{h} \perp 0z$ then P can be considered as an inner point.

Equations (4.1.33),(4.1.34),(4.1.35) can be written:

$$U\ FD\{\xi D_{xx}U\} = -UF\ V\{\xi(D_x U)^2\}$$

$$U\ FD\{\xi D_{zz}U\} = -UF\ V\{\xi(D_z U)^2\}$$

$$\forall P\ \varepsilon\ \overline{\Omega} = (\Omega + \Gamma) \qquad (4.1.36)$$

$$U\ FD\{\xi D_{xz}W\} = -UF\ V\{\xi D_x W D_z U\}$$

$$\forall P\ \varepsilon\ \Omega\ (P\ \text{inner point})$$

and $$P \not\subset \Gamma \qquad (4.1.37)$$

$$U\ FP\{\xi D_{xz}W\} \neq -UF.V\{\xi D_x W D_z U\}$$

$$\forall P\ \varepsilon\ \Gamma\ (P\ \text{Boundary point}) \qquad (4.1.38)$$

since we have $\Omega \subset \mathbb{R}^2 \times [OT]$

in the absence of body forces one can write:

$$-\int_\Omega F\ \mathbf{U}\ d\Omega = \int_\Omega \rho D_{tt}U\ \mathbf{U}\ d\Omega \qquad (4.1.39)$$

i.e.

$$- \int_{\Omega_h} F d\Omega = \int_{\Omega_h} \rho (D_t U)^2 d\Omega_h$$

$$= -\frac{1}{\Delta t} U_0 [U^1 - 2U + U^{-1}] \{\text{Mass}\}_{\Omega_h} \qquad (4.1.40)$$

4.1.d Recursive Developments

Replacing in equations (4.1.10), (4.1.11) the respective expressions by their FV{.} developments and taking into account (4.1.10), we obtain

$$U^1 = 2U - U^{-1} + \frac{2\Delta t^2}{h_1^2[\rho_1+\rho_3]}[[U_1 - U_0](\lambda+2\mu)_1$$

$$- (U_0 - U_3)(\lambda+2\mu)_3]$$

$$+ \frac{2\Delta t^2}{h_2^2[\rho_2+\rho_4]}[(U_2 - U_0]\mu_2 - [U_0 - U_4)]\mu_4]$$

$$+ \frac{\Delta t^2}{4h_1h_2(\rho_3+\rho_1)} F\,V\{\lambda D_z W D_x U\}$$

$$+ \frac{\Delta t^2}{4h_1h_2(\rho_2+\rho_4)} F\,V\{\mu D_x W D_z U\} \qquad (4.1.41)$$

$$W^1 = 2U - U^{-1} + \frac{2\Delta t^2}{h_2^2(\rho_2+\rho_4]}[(W_2 - W_0)(\lambda+2\mu)_2$$

$$- (W_0 - W_4)(\lambda+2\mu)_4]$$

$$+ \frac{2\Delta t^2}{h_1^2(\rho_1+\rho_3)} [(W_1 - W_0]\mu_1 - (W_0 - W_3)\mu_3]$$

$$+ \frac{\Delta t^2}{4h_1h_2[\rho_4+\rho_2]} F\ V[\lambda D_x U D_z W]$$

$$+ \frac{\Delta t^2}{4h_1h_2[\rho_3+\rho_1]} F\ V[\mu D_z U D_x W] \qquad (4.1.42)$$

By comparison with equations (3.4.1) and (3.4.2) we see that the two systems are identical.

4.1.e Uniqueness

As a consequence of the identity of the development between the variational method and by Taylor expressions we are led to the following proposition:

The elastic wave equation admits an identical second order numerical expression if the trial function has the same norm in the conservation of energy space W_2^1.

4.2 CASE OF EQUATIONS OF COMPRESSIONAL AND ROTATIONAL DEFORMATIONS

Let us write the equations of motion for a homogeneous, isotropic elastic media

$$(\lambda + 2\mu)\nabla\nabla U - \mu\nabla\wedge\nabla\wedge U = \rho\ddot{U} - \rho K \qquad (4.2.1)$$

If we take successively the divergence and the curl of equation (4.2.1), the dilatation

$$P = \text{div } U \qquad (4.2.2)b$$

and the vector rotation

$$S = \text{curl } U \qquad (4.2.2)c$$

satisfy the respective equations

$$(\lambda + 2\mu)\nabla^2 P = \rho\ddot{P} - \rho\nabla.K \qquad (4.2.2)$$

$$\mu\nabla^2 S = \rho\ddot{S} - \rho\nabla\wedge K \qquad (4.2.3)$$

where K represents the body forces.

The aim of this section is to give a finite development for those equations by the two methods developed in these pages. i.e. The finite development by variational principle, and the generalised solution of the direct Taylor development.

4.2.a Development by Variational Method

If we consider the usual elementary domain $D = \Omega_h$, the respective scalar product of (4.2.2), (4.2.3) with P and S yields in the absence of body forces

$$\int_{\Omega_h} \alpha^2 (\nabla P)^2 d\Omega_h - \int_{\Gamma_h} \alpha^2 \nabla P d\Gamma = \int_{\Omega_h} (\dot{P})^2 d\Omega_h \qquad (4.2.4)$$

$$\int_{\Omega_h} \beta^2 (\nabla S)^2 d\Omega_h - \int_{\Gamma_h} \beta^2 \nabla S d\Gamma = \int_{\Omega_h} (\dot{S})^2 d\Omega_h$$

$$S, P \in W_0^1 \subset L^2(\Omega)$$

with $$\overline{\Omega}_h = \Omega_h + \Gamma_h \subset \Omega \subset R^2 \times [0T] \qquad (4.2.5)$$

where $$\beta^2 = \mu/\rho$$

$$\alpha^2 = (\lambda + 2\mu)/2$$

(4.2.4), (4.2.5) yield from section 4.1

$$\int_{\Omega_h} PF\ V\{\alpha^2(\nabla P)^2\} - \int_{\Gamma_h} P\alpha^2 \nabla P d\Gamma_h = \int_{\Omega_h} PF\ V[D_t P]^2 d\Omega_h \quad (4.2.6)$$

$$\int_{\Omega_h} SF\ V\{\beta^2(\nabla S)^2\} - \int_{\Gamma_h} S\beta^2 \nabla S d\Gamma_h = \int_{\Omega_h} SF\ V[D_t S]^2 d\Omega_h$$

$$\text{in } \overline{\Omega}_h \subset R^2 \times [OT] \quad (4.2.7)$$

The boundary conditions of (4.2.6) can be written

$$\int_{\Gamma} \alpha^2 \nabla P d\Gamma = \Delta t h_2[\alpha_1^2 P_x - \alpha_3^2 P_x] + h_1[\alpha_2^2 P_z - \alpha_4^2 P_z]\Delta t \quad (4.2.8)$$

where α_1^2, α_2^2 represents the values of the parameters in the usual domains, i.e. $D^I \cup D^{IV}$, etc.

Since

$$P = U_x + W_z \quad (4.2.9)$$

$$S = U_z - W_x \quad (4.2.10)$$

The stress continuity conditions can be written

$$(\alpha^2 P - 2\beta^2 W_z)_1 = (\alpha^2 P - 2\beta^2 W_z)_3 \qquad (4.2.11)$$

$$(\alpha^2 P - 2\beta^2 U_x)_2 = (\alpha^2 P - 2\beta^2 U_x)_4 \qquad (4.2.12)$$

$$[\beta(S - 2U_z)]_2 = [\beta(S - 2U_z)]_4 \qquad (4.2.13)$$

$$[\beta(S - 2W_x)]_1 = [\beta(S - 2W_x)]_3 \qquad (4.2.14)$$

The expressions (4.2.13) and (4.2.14) being identical.

After differentiating (4.2.11) and (4.2.14) respectively with regard to the variable x and z, we obtain

$$\alpha_1^2 P_x - \alpha_3^2 P_x = \beta_3^2 S_z - \beta_1^2 S_z \ . \qquad (4.2.15)$$

In the same way taking into account (4.2.11) and (4.2.14)

$$\alpha_2^2 P_z - \alpha_4^2 P_z = \beta_2^2 S_x - \beta_4^2 S_x \qquad (4.2.16)$$

the boundaries concerning equation (4.2.7) yields

$$\beta_1^2 S_x - \beta_3^2 S_x = \frac{\lambda_1}{\rho} P_z - \frac{\lambda_3}{\rho} P_z \qquad (4.2.17)$$

$$\beta_2^2 S_z - \beta_4^2 S_z = \frac{\lambda_4}{\rho} P_x - \frac{\lambda_2}{\rho} P_z \qquad (4.2.18)$$

with $$\frac{\lambda}{\rho} = \alpha^2 - 2\beta^2$$

After inserting the boundary conditions in the equations (4.2.6), (4.2.7) and replacing the values of $F_V\{\cdot\}$ by their developments (section 4.1) we obtain

$$P^1 = 2P - P^{-1} + \frac{1}{h_1^2}\left[(P_1 - P_0)\alpha_1^2 - (P_0 - P_3)\alpha_3^2\right]$$

$$+ \frac{\Delta t^2}{h_2^2}\left[(P_2 - P_0)\alpha_2^2 - (P_0 - P_4)\alpha_4^2\right]$$

$$+ \frac{\Delta t^2}{h_1}\left[\beta_1^2 S_z - \beta_3^2 S_z\right]$$

$$+ \frac{\Delta t^2}{h_2}\left[\beta_4^2 S_x - \beta_2^2 S_x\right] \qquad (4.2.19)$$

$$S^1 = 2S - S^{-1} + \frac{\Delta t^2}{h_1^2}\left[(S_1 - S_0)\beta_1^2 - (S_0 - S_3)\beta_3^2\right]$$

$$+ \frac{\Delta t^2}{h_2^2}\left[(S_2 - S_0)\beta_2^2 - (S_0 - S_4)\beta_4^2\right]$$

$$+ \frac{\Delta t^2}{h_2}\left[\lambda_2 P_x - \lambda_4 P_x\right] + \frac{1}{h_1}\left[\lambda_3 P_z - \lambda_1 P_z\right] \qquad (4.2.20)$$

We note that the boundary terms

$$\frac{1}{h_1}(\beta_1^2 S_z - \beta_3^2 S_z),\ \frac{1}{h_2}(\beta_4^2 S_x - \beta_2^2 S_x),$$

$$\frac{1}{h_2}(\lambda_2 P_x - \lambda_4 P_x),\ \frac{1}{h_1}(\lambda_3 P_z - \lambda_1 P_z) \tag{4.2.21}$$

represent the conversion terms SV to P and P to SV.

The finite development of those terms have the form

$$\frac{1}{h_1}(\beta_1^2 S_z - \beta_3^2 S_z) \simeq \frac{1}{2h_1 h_2}(\beta_1^2 - \beta_3^2)(S_2 - S_4) \tag{4.2.22}$$

$$\frac{1}{h_2}(\beta_4^2 S_x - \beta_2^2 S_x) \sim \frac{1}{2h_1 h_2}(\beta_4^2 - \beta_2^2)(S_1 - S_3) \tag{4.2.23}$$

4.2.b Development by Taylor Expansion

Equation (4.2.2) and (4.2.3) yield

$$\text{F.D.}[\alpha^2\nabla^2 P] - [\alpha_1^2 - \alpha_3^2]\tilde{D}_x P$$

$$- (\alpha_2^2 - \alpha_4^2)\tilde{D}_z P = \text{FD}(\ddot{P}) \tag{4.2.24}$$

$$\text{FD}[\beta^2\nabla^2 S] - (\beta_1^2 - \beta_3^2)\tilde{D}_x S$$

$$- (\beta_2^2 - \beta_4^2)\tilde{D}_z S = \text{F.D}(\ddot{S}) \tag{4.2.25}$$

Since the conversion terms are identical to expressions (4.2.15) to (4.2.18) the development leads to:

$$F\ D\{\nabla^2\} \equiv FV\{(\nabla)^2_{\cdot}\}$$

The development of equations (4.2.25) and (4.2.26) is identical to equations (4.2.19) and (4.2.20) obtained by variational principle. i.e. After taking into account the boundary conditions (4.2.11) to (4.2.14), (4.2.25) and (4.2.26) yield

$$FD(\ddot{P}) = F\ D[\alpha^2\nabla^2 P] + \beta^2_x\tilde{S}_z - \beta^2_z\tilde{S}_x \qquad (4.2.26)$$

$$FD(\ddot{S}) = F\ D[\beta^2\nabla^2 S] + \frac{\lambda}{\rho}_z\tilde{P}_x - \frac{\lambda}{\rho}_x\tilde{P}_z \qquad (4.2.27)$$

which is identical to (4.2.19) and (4.2.20) or in the form

$$\tilde{\ddot{P}} = \alpha^2\tilde{\nabla}^2 P + \alpha^2_x\tilde{P}_x + \alpha^2_z\tilde{P}_z + \beta^2_x S_z - \beta^2_z\tilde{S}_x \qquad (4.2.28)$$

$$\tilde{\ddot{S}} = \beta^2\tilde{\nabla}^2 S + \beta^2_x\tilde{S}_x + \beta^2_z\tilde{S}_z + \frac{\lambda}{\rho}_z\tilde{P}_x - \frac{\lambda}{\rho}_x\tilde{P}_z \qquad (4.2.29)$$

where $\sim$ represents the homogeneous development of section 2.4. We have expressed the form (4.2.28) and (4.2.29) not in a computational form but for qualitative purposes.

4.2.c Reflection and Conversion Terms Considered as Source

We notice that in equation (4.2.28) for instance $\alpha_x^2 P_x$, $\alpha_z^2 P_z$ represents the reflection and refraction terms.

$(\beta_x^2 S_z - \beta_z^2 S_x)$ represents the SV→P conversion term

$(\frac{\lambda}{\rho}_z \tilde{P}_x - \frac{\lambda}{\rho}_x \tilde{P}_z)$ represents the P→SV conversion terms.

From (4.2.28) and (4.2.29) we note the symmetry of reflection terms and conversion terms.

Besides, equation (4.2.29) shows that even in a homogeneous fluid SV may be generated but does not propagate; since (4.2.27) can be written

$$\ddot{S} = \text{curl } K \qquad (4.2.30)$$

with
$$\text{curl } K = \lambda_z P_x - \lambda_x P_z \qquad (4.2.31)$$

But
$$S(M) \equiv 0 \quad \forall\ F(M) = 0$$
$$\forall\ M \in \Omega \subset \mathbb{R}^2 \times [OT]$$

In (4.2.30) we have considered the transformation of a source, i.e. as a body force. Then in equations (4.2.26), (4.2.27), the reflection and transmission terms can be considered as sources (or body forces) due to a discontinuity (parameter discontinuity), those body forces, satisfy

of course the B.C. Then from (4.2.26) and (4.2.27) we can write the reflection and conversion terms considered as body forces

$$\nabla . K = \alpha_x^2 P_x + \alpha_z^2 P_z + \beta_x^2 S_z - \beta_z^2 S_x \qquad (4.2.32)$$

$$\nabla \wedge K = \beta_x^2 S_x + \beta_z^2 S_z + \frac{\lambda}{\rho}{}_z P_x - \frac{\lambda}{\rho}{}_x P_z \qquad (4.2.33)$$

By analogy with the equation of displacement motion we can write Kx, Kz as the components of the reflection and conversion coefficients

$$Kx = \alpha_x^2 U_x + \frac{\lambda}{\rho}{}_x W_z + \beta_z^2 U_z + \beta_z^2 W_x \qquad (4.2.33)$$

$$Kz = \alpha_z^2 W_z + \frac{\lambda}{\rho}{}_z U_x + \beta_x^2 W_x + \beta_x^2 U_z \qquad (4.2.34)$$

But $$\text{curl } K = Kx_z - Kz_x$$

Expressing curl K with the help of (4.2.33) and (4.2.34), and taking into account that

$$P = U_x + W_z$$

$$S = U_z - W_x$$

as well as the B.C

$$\sigma_{xz}\Big|_{-}^{+} = 0$$

we obtain

$$\text{curl } K = \beta_x^2 S_x + \beta_z^2 S_z + \lambda_z P_x - \lambda_x P_z$$

which is the relation (4.2.33), which can be considered as a verification of the assertions given in this section.

4.3 "HETEROGENEOUS MEDIA SOLUTION" OF EQUATIONS OF COMPRESSIONAL AND ROTATIONAL DEFORMATION

4.3.a Development of the so called "heterogeneous media solution"

Following Landers and Claerbout (1972), Grant and West (1965), we consider the law of motion for the displacement vector U, i.e.

$$\rho\ddot{U} = (\lambda + \mu)\nabla(\nabla.\vec{U}) + \nabla\lambda\nabla.\vec{U}$$

$$- \mu\nabla\wedge\nabla\ \vec{U} + \nabla\mu\wedge(\nabla\wedge\vec{U})$$

$$+ 2(\nabla\mu\nabla)\vec{U} \qquad (4.3.1)$$

(Karal and Keller, 1959)

Letting $\nabla.\vec{U} = P$

$$\nabla\wedge\vec{U} = S$$

We obtain after taking the div. and curl of (4.3.1)

$$\ddot{P} = \alpha^2[P_{xx} + P_{zz}) + 2\alpha_x^2 P_x + 2\alpha_z^2 P_z$$

$$+ 2[\beta_x^2 S_z - 2\beta_z^2 S_x] \qquad (4.3.2)$$

$$\ddot{S} = \beta^2[S_{xx} + S_{zz}) + 2\beta_x^2 S_x + 2\beta_z^2 S_z$$

$$+ [2\beta_z^2 S_x - 2\beta_x^2 S_z] \qquad (4.3.3)$$

(Landers and Claerbout, 1972)

4.3.b Remarks

We see that if the "heterogeneous media" method is correct we shall have two important properties

i) From (4.3.2) and (4.3.3) the conversion terms are independent of λ, i.e. waves can be converted only if there is a discontinuity in shear velocity.

ii) If we consider a fluid equation (4.3.2) becomes

$$\ddot{P} = \alpha^2[P_{xx} + P_{zz}] + 2[\alpha_x^2 P_x + \alpha_z^2 P_z]$$

instead of

$$\ddot{P} = \alpha^2[P_{xx} + P_{zz}] + \alpha_x^2 P_x + \alpha_z^2 P_z$$

i.e., the reflection coefficients will be twice the reflection coefficients of a scalar wave equation or a sound wave. Therefore the system (4.3.2), (4.3.3.) is obviously not correct.

4.4 General of P→SV, SV→P converted waves

Elastic wave motion is governed by equations (4.2.29), (4.2.3), i.e.

- variation in λ generate P→SV
- variation in μ generate SV→P.

Figures (4.1a) to (4.1d) represent snap shots of P→SV converted wave (P source) for a simple model, (Fig. 4.1); the source having a Gaussian dependence in a domain of definition $\Omega_s = P \pm 2h \subset R^2 \times [0,t)$ and where the spatial dependence S(X) of the source trends smoothly to zero at the boundary of its domain of definition.

The model consists of two layers of 7000'/sec and 10,000'/sec. The wave field computed with a coarse grid is represented for qualitative purpose and shows the relative importance of converted shear wave.

Fig. 4.1c represents the shear wave field, and comparison with the P→SV reflection coefficient curve as a function of the angle of incidence shows the correlation between the theoretical curve and the synthetic wave field.

REFLEXION AND REFRACTION OF A COMPRESSIONAL WAVE

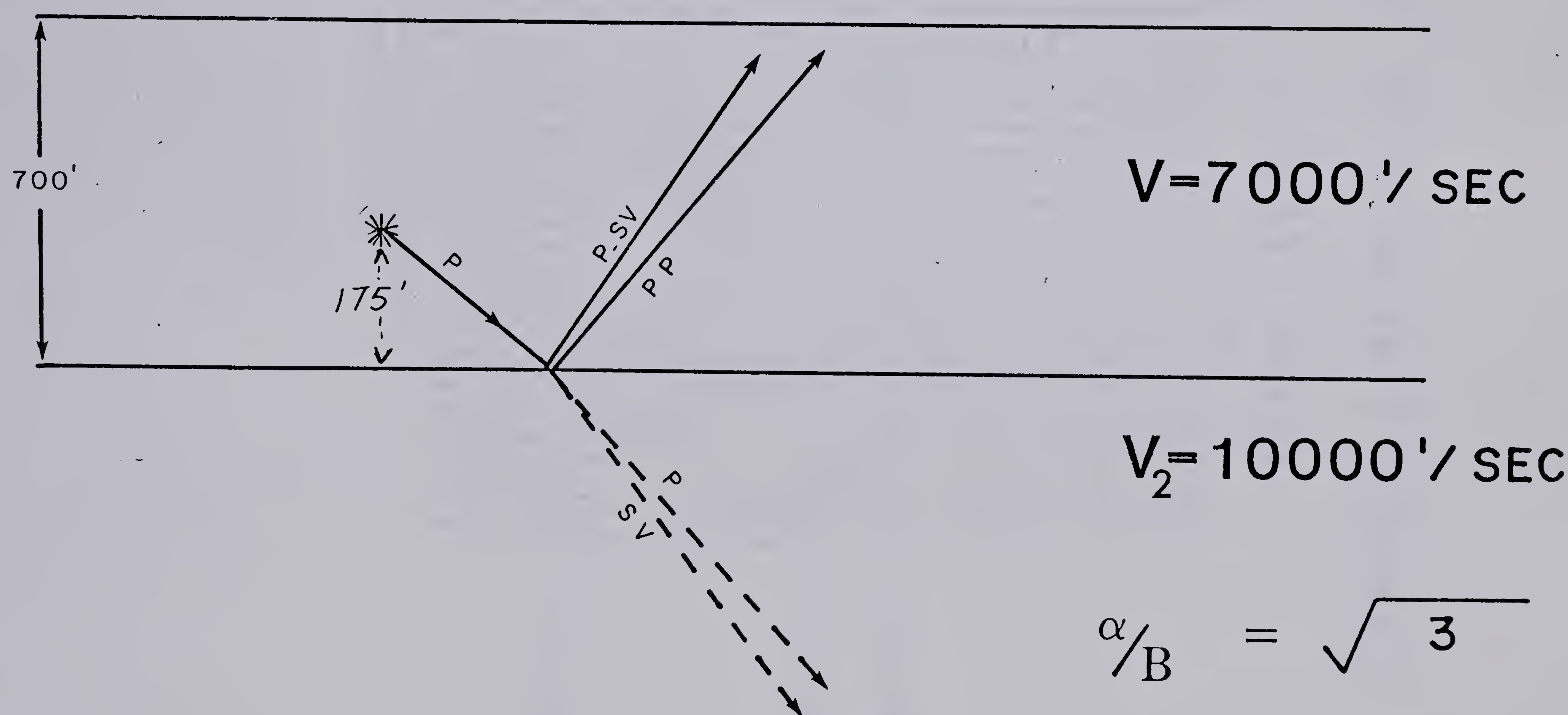


Fig. 4.1. Reflection and conversion of a compressional wave
4.1.a One layer model

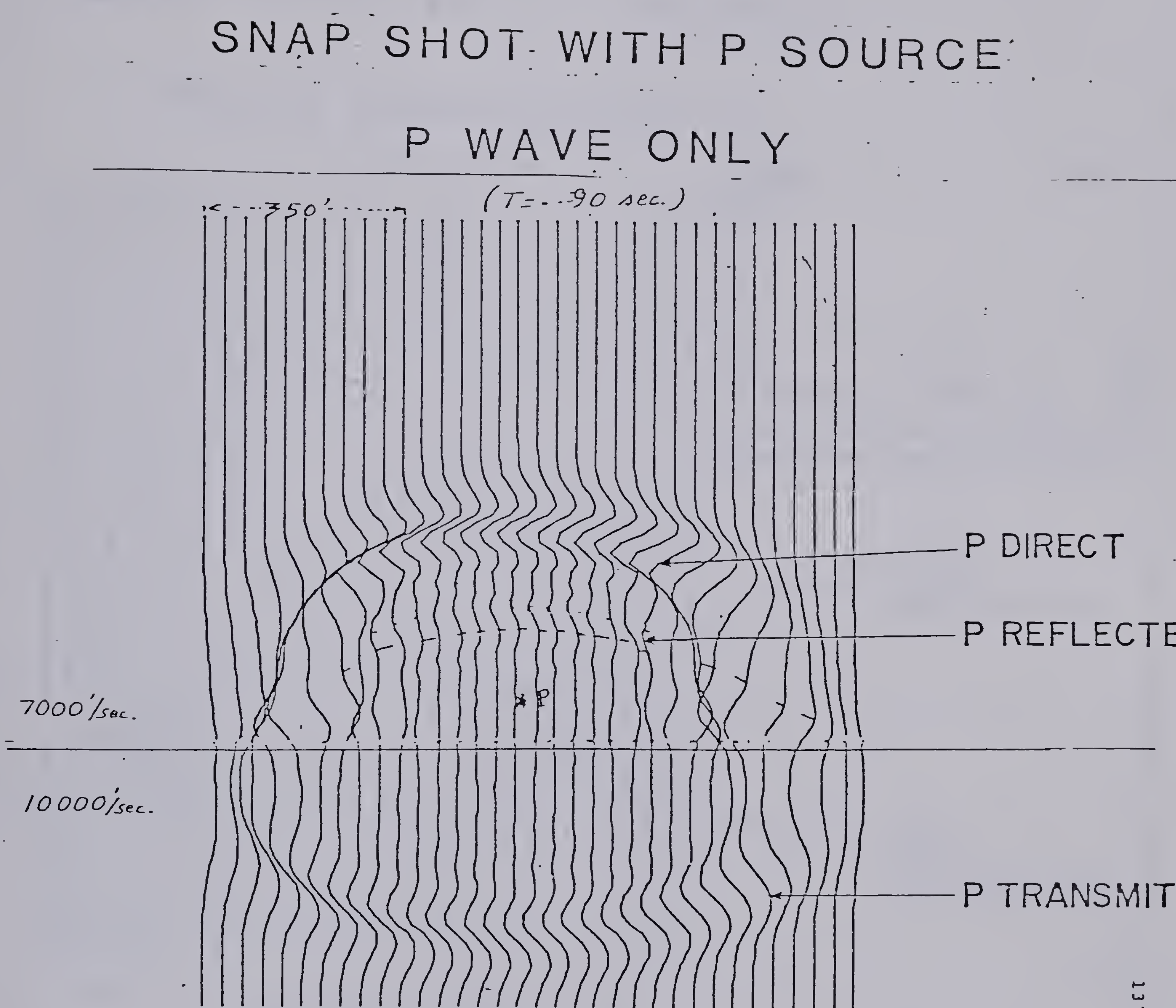


Fig. 4.1.b Wave field of reflected and transmitted P wave

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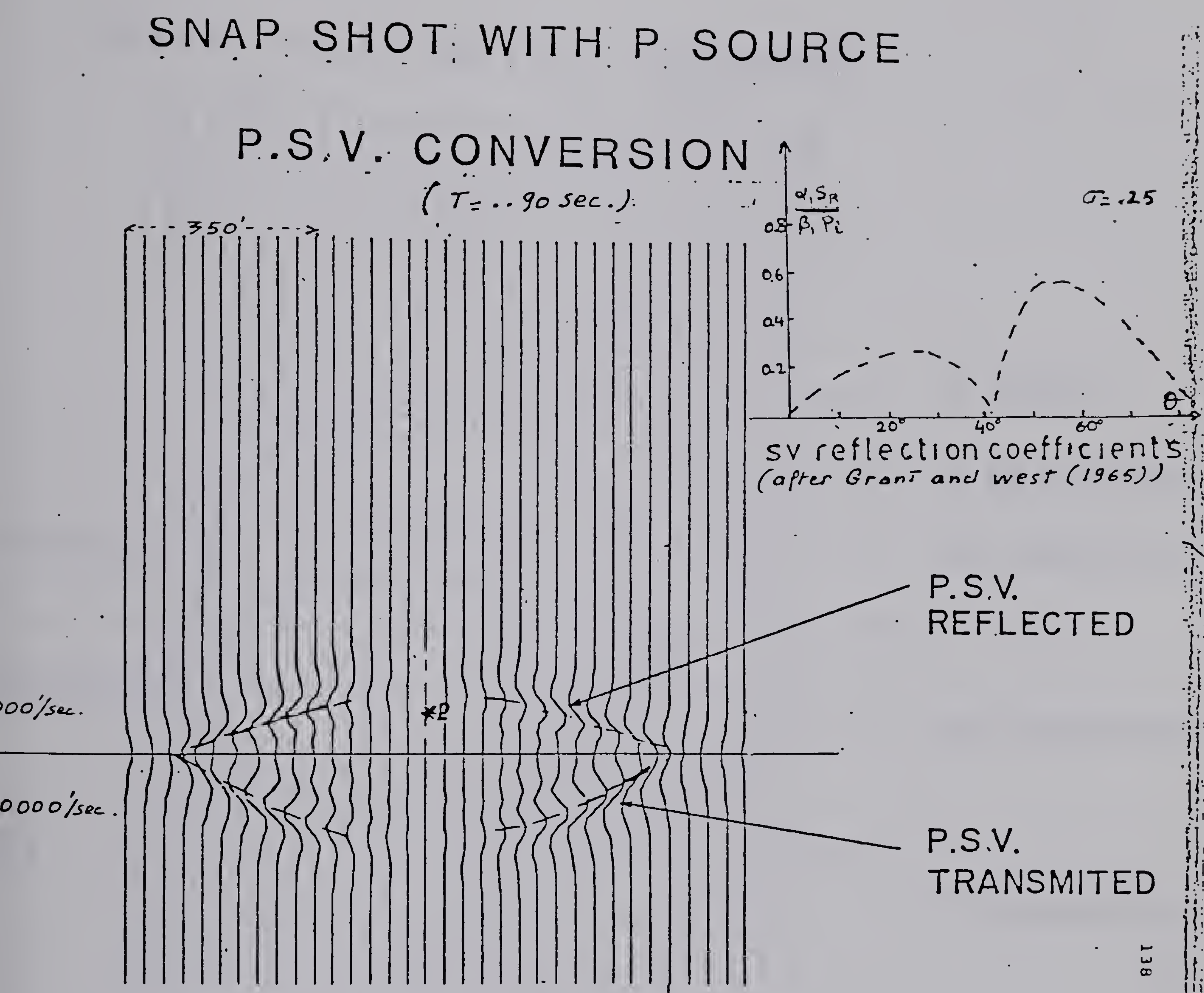

Fig. 4.1.c Wave field of converted waves (P→SV)

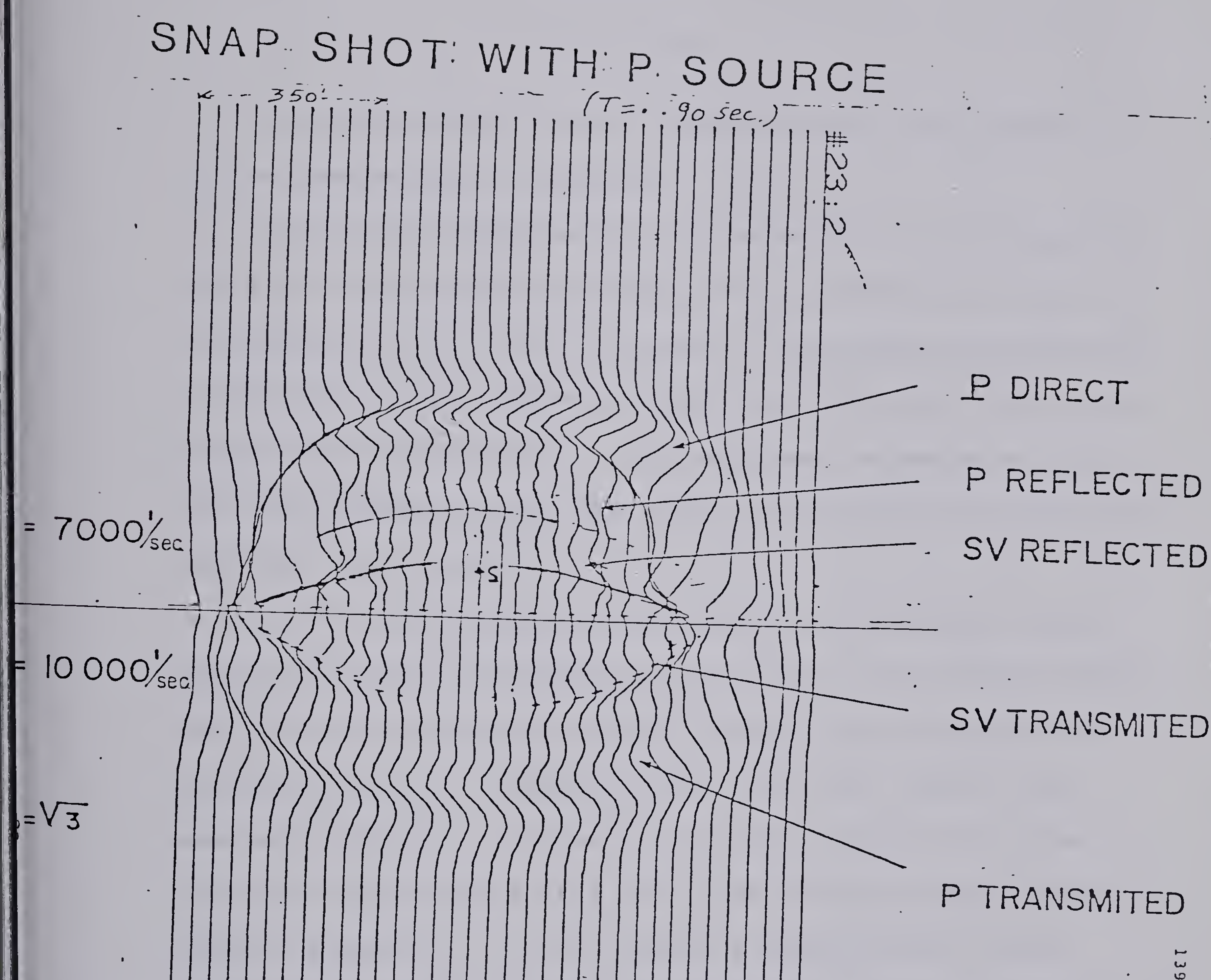

Fig. 4.1.d Total wave field.

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CHAPTER V

Conclusion

The main points of this research which are original can be summarized as follows:

i) The distribution of the parameters at the boundary. Since we can consider a measure as the inverse of a partial differential (D_j^{-1}), it is normal to consider the parameters as specific to the operator domain (in this case the domain of the second derivative). An important consequence being that the parameters can vary with the direction and diagonal boundary can be used.

ii) The difficulty of introducing the boundary conditions inside the differential equation for the direct method has been solved simply by adapting the Taylor expansion to the specificity of the parameters. We note that if the boundary conditions do not vanish when introduced in the differential equation, the so called "heterogeneous media" method (Kelly et al, 1976) cannot be used, which is the case of most problems except in the SH or the scalar wave equation. The direct scheme is valid for any case of boundary conditions and can be considered as a direct generalised scheme for heterogeneous media.

iii) The Dirichlet integral and the energy method give us a classical way of introducing the boundary conditions. The z transform and the choice of a trial function which has the same norm as for the direct method (in Sobolev W_2^1 space which is a Hilbert space) allow us to obtain a finite development which is comparable to the direct form. The identity of the schemes and their uniqueness in the two methods developed shows the problem is well posed.

iv) The P SV, and SV P transformations have been studied through the divergence and rotation or curl of the displacement vector equation. The direct and the finite variational developments have been successively applied and lead to identical schemes. Those solutions correlate with the displacement vector solutions seen above. The parameter discontinuity is considered as a source of the scattered and converted fields in the displacement vector equation. The divergence and curl of those forces give the scattered and converted field in terms of the divergence and curl of the vector displacement equations. In this case als the comparison with the so called "heterogeneous media" method, gives the same conclusion as before. Although the results are not concordant to the one obtained by Landers et al (1972), and Grant and West (1965), the evidence led us

to believe that the SV→P and P→SV conversions are given respectively by discontinuity of the parameters μ and λ (and not only μ as stated). This result is important:

- as an interpretative tool in seismic exploration.
- to investigate the bright spot problem (Stoffa et al, 1976).

v) The secondary results of this research are:

- the representations of the source as the causal part of the development of a force. This underlines the reciprocal properties of the wave equation and is very useful in the numerical solution.
- The finite difference development of the SH or scalar wave equations at normal incidence for $\frac{V\,t}{h} = 1$ is an exact solution. That is, the simple finite difference development is identical to the analytical plane wave solution or the matrix method.

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